UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Physics 1B46: Mathematical Methods II

COURSE CODE	:	**PHYS1B46**
UNIT VALUE	:	**0.50**
DATE	:	**18-MAY-06**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours 30 Minutes**

06-C1159-3-130

TURN OVER

Answer **FIVE** *questions.*

A formula sheet giving the Lorentz transformation is attached after the end of the paper.

Numbers in square brackets show the provisional allocation of marks per question section.

1 Consider the following differential equations:

$$\frac{dy}{dx} = -ky \quad (1)$$

$$y\frac{dy}{dx} = 4x(y+1)+2 \quad (2)$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = \cos x \quad (3)$$

$$y\frac{dy}{dx} = 2(x+1)/y \quad (4)$$

1(a) Which of these is a linear, homogeneous first-order differential equation? **[1 mark]**
Write down the most general solution to this equation, and give the particular solution which satisfies the boundary condition $y = 1$ when $x = 0$. **[3 marks]**

1(b) Which of these is a separable first order differential equation, but not linear? **[1 mark]**

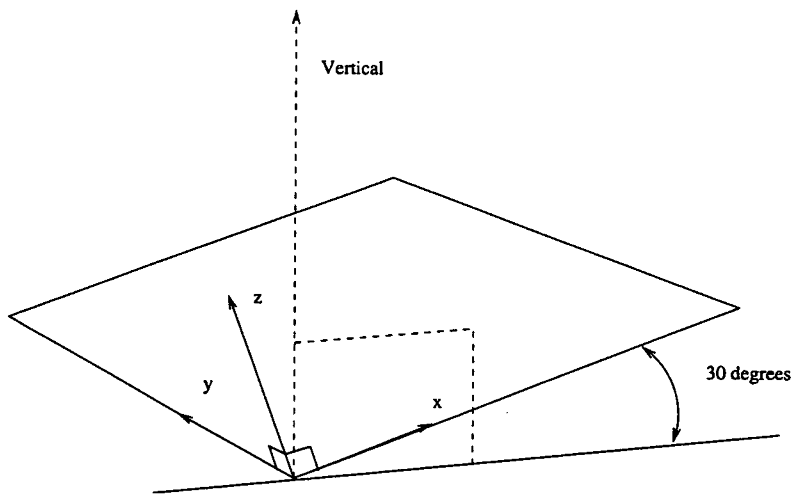
Find the general solution to this equation, as well as the particular solution satisfying the boundary condition $y = 1$ when $x = 0$. **[5 marks]**

1(c) Which of the above equations is a linear second order differential equation with constant coefficients? **[1 mark]**

By choosing an appropriate trial solution, find a function which satisfies the equation (i.e. the particular integral). **[5 marks]**

Find the complementary function and combine this with the particular solution you have already found, to obtain the general solution of the differential equation. **[4 marks]**

2(a) A mountainside is approximated by the x, y plane where the coordinates x, y and z define a right-handed system with z pointing upward out of the plane. The x axis of the plane is inclined at 30^o to the horizontal and the bottom of the plane coincides with the y axis, which is horizontal (see diagram).



Consider a child on skis, with total mass m, on the mountainside. Taking the zero point of gravitational potential to be the origin, give an expression for the gravitational potential energy $V(x, y)$ of the child on the plane. **[4 marks]**

2(b) Give an expression for the gravitational force F on the child in terms of V, and evaluate this as a function of cartesian unit vectors. **[5 marks]**

2(c) The child has very good skis, so there is no friction. The child is pushed up from the point $(0, 0)$ to the point $(1, 1)$ along the path defined by $x = y$. What is the work done on the child in moving him? What is the change in potential energy of the child? **[4 marks]**

2(d) The child has an accident on the way down and damages a ski so that there is now a frictional force. The child is pushed back up the slope in the same path as before, from $(0, 0)$ to $(1, 1)$, such that the frictional force is constant and equals $\mathbf{f} = -A(\mathbf{i} + \mathbf{j})$. What is the total work done on moving child this time? What is the change in potential energy of the child? **[7 marks]**

3(a) Calculate the numerical value of the area integral:

$$I = \int_0^1 dx \int_0^x dy\, xy^2 \, .$$

and sketch the region over which the integral is carried out. What value would you calculate for the integral

$$I = \int_{-1}^1 dx \int_0^x dy\, xy^2 \, .$$

[6 marks]

3(b) With the aid of a sketch, explain how the position of a point in three-dimensional space can be specified using cylindrical polar coordinates r, θ, z. Show the volume element and give an expression for it.
Give expressions for the Cartesian coordinates x, y and z in terms of r, θ and z. **[4 marks]**

3(c) Calculate the numerical value of the volume integral

$$I = \int P\, dv = \int (x^2 + y^2 + z^2)\, dv$$

over a cylinder of unit radius and unit height centred on the origin. **[6 marks]**

3(d) Interpreting P as a scalar field, calculate the gradient of P for any point within the cylinder in terms of x, y and z.
In which direction does P increase? Where in the cylinder or on its edge does P have its maximum and minimum values? **[4 marks]**

4(a) The points (0,0,0), (0,1,0), (1,1,0) and (1,0,0) define a square in the x, y plane. What is the flux of the vector field $\mathbf{F} = x\mathbf{j} + 2\mathbf{k}$ through the interior of the square? **[4 marks]**

4(b) Calculate the curl of the field, $\nabla \times \mathbf{F}$.
Evaluate the area integral $\int (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ over the square. State the relationship between this area integral and the line integral $\oint \mathbf{F} \cdot d\mathbf{s}$ around the perimeter of the square. **[5 marks]**

4(c) The matrix

$$R = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

describes a rotation about the y axis by angle θ. Give the coordinates of the corners of the square after applying this transformation. Give a vector normal to the rotated square. **[4 marks]**

4(d) Calculate the flux of the field F though the rotated square. **[3 marks]**

4(e) Calculate the determinant of the matrix and use this to calculate the matrix for the inverse transformation. **[4 marks]**

5(a) A particle with rest mass m_0 travels with velocity $\mathbf{v}$ in some inertial frame S. State the formulae for the 3-momentum $\mathbf{p}$ and the energy E in terms of m_0 and $\mathbf{v}$. **[2 marks]**

5(b) If $\mathbf{v}$ is along the x axis, write down the energy momentum four-vector p for the particle in terms of the 3-momentum $\mathbf{p}$, the energy of the particle, E, and the speed of light, c. Calculate the magnitude (length) of the four-vector and use the expressions you gave in the first part to show that it is a Lorentz invariant (i.e. independent of $v = |\mathbf{v}|$). **[4 marks]**

5(c) In its rest frame, S', the particle spontaneously decays at rest into two identical massless particles, which emerge along the positive and negative x-axes. Use conservation of four-momentum to obtain an expression for the four momentum of each of these particles in S' terms of m_0. **[5 marks]**

5(d) Apply the Lorentz matrix transformation to the four-vectors of these particles to obtain their four-momenta in S. **[4 marks]**

What is the speed of the two massless particles in the frame S? **[2 marks]**

5(e) If the mean decay lifetime of the original particle in its rest frame S' is τ, what is its mean lifetime as observed in the frame S? **[3 marks]**

6(a) Write down the energy-momentum four-vector for a photon in terms of its energy E and linear 3-momentum $\mathbf{p} = (p_x, p_y, p_z)$, and the speed of light, c. What is the length of the four vector? Use this to obtain a relationship between E and $|\mathbf{p}|$. **[4 marks]**

6(b) State the relationship (de Broglie formula) between the momentum $\mathbf{p}$ of a photon and its wavelength, λ. Use this and the result derived above to express the energy-momentum four-vector of a photon in terms of λ. **[3 marks]**

6(c) A star is receding from the Earth with a speed $v = 0.5c$, and emitting light which has wavelength λ_0 in its rest frame. The light is observed by astronomers on Earth to have a wavelength $\lambda = 800\text{nm}$. By applying the Lorentz tranformation to the four-vector derived above, or by other methods, calculate λ_0. **[6 marks]**

6(d) A second star, of the same type, travels in a direction transverse to the line of observation from Earth, with a speed $0.5c$. It also emits light with wavelength λ_0 in the rest frame of the star. Calculate the wavelength of the light as observed on Earth, and also the angle at which the light is emitted in the rest frame of the star. Comment briefly on how your result relates to relativistic time dilation. **[7 marks]**

7(a) A pair of unbiased dice have six sides each numbered 1 to 6. They are rolled ten times. Calculate the probability of rolling a total score $s = 8$ on the first roll. Calculate the probability of coming up with a score of eight for all ten rolls. The first nine rolls do in fact come up with eight. What is the probability that the tenth roll comes up with eight? **[3 marks]**

7(b) A discrete random variable x has N possible values, x_i, $i = 1 \rightarrow N$, each with probability p_i. Give the formula for the mean value and standard deviation of x in terms of p_i, x_i and N. Calculate the mean and standard deviation for the total score s on the two dice. **[5 marks]**

7(c) A Higgs particle is produced at rest in a collider experiment and decays at the origin to two particles which travel back-to-back in a direction specified by the angles θ and ϕ in spherical polar coordinates. The particles are detected in a spherical detector centred on the origin. The probability of the particles hitting a point on the surface of the detector is independent of the position. That is, the probability of the particle hitting an solid angle element $\sin\theta d\theta d\phi$ is

$$P(\theta,\phi)\sin\theta d\theta d\phi = A\sin\theta d\theta d\phi$$

where A is a constant.

Evaluate A, and give an expression for the probability of observing the particle at an angle between θ and $\theta + d\theta$. **[4 marks]**

7(d) The transverse momentum of a particle produced in the decay is $62\sin\theta$ (in GeV/c). Calculate the mean transverse momentum $\langle p_T \rangle$ observed in an experiment which produces many such Higgs particles. **[4 marks]**

7(e) The detector cannot see the particles if $\theta < \pi/6$ or $\theta > 5\pi/6$. Calculate the probability that a particular decay is observed in the detector. **[4 marks]**

END OF PAPER
FORMULA SHEET ATTACHED

Formula Sheet

The Lorentz transformation relationing a four-vector defined in frame S to that defined in the frame S' which is moving along the x axis of S with speed v is

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\frac{v}{c} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

where c is the speed of light, and γ is defined as:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} .$$

For the position four-vector, $a = x, b = y, c = z, d = ct$, and for the energy momentum four-vector, $a = p_x, b = p_y, c = p_z, d = E/c$.

The metric in any inertial frame is:

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

END OF FORMULA SHEET