UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. M.Sci.

Physics 1B46: Mathematical Methods II

COURSE CODE	:	**PHYS1B46**
UNIT VALUE	:	**0.50**
DATE	:	**19-MAY-05**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours 30 Minutes**

05-C1075-3-160

TURN OVER

Answer **FIVE** *questions.*

A formula sheet giving the equations of the Lorentz transformation is attached after the end of the paper.

Numbers in square brackets show the provisional allocation of marks per question section.

1(a) What is the most general solution of the linear homogeneous first-order differential equation

$$\frac{dy}{dx} = -ky ,$$

and what is the solution that satisfies the boundary condition $y = 2$ when $x = 0$?

[3 marks]

Name a physical process that is described by a first-order differential equation having the above form, and state the physical meaning of the dependent and independent variables in this process. **[2 marks]**

1(b) Use the "integrating factor" method to find the general solution of the differential equation

$$\frac{dy}{dx} + \frac{1}{1+x}y = x ,$$

noting that your general solution should contain one arbitrary constant. **[6 marks]**

1(c) Determine the coeffients α and β such that $y = \alpha \sin x + \beta \cos x$ is a particular integral of the second-order differential equation:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \cos x .$$

[4 marks]

By looking for a complementary function of the form $y = \exp(px)$, obtain the general solution of the differential equation. **[3 marks]**

Find the values of the arbitrary constants in the general solution for which $y = 1$ and $dy/dx = 1$ when $x = 0$. **[2 marks]**

2(a) Calculate the value of the volume integral

$$I = \int\int\int r^2 \, dx \, dy \, dz$$

over the interior of the cube whose centre is at the origin and whose edges are parallel to the x, y and z-axes and have length 2. (The quantity $r^2 = x^2 + y^2 + z^2$ is the square of the distance from the origin.) **[5 marks]**

2(b) Explain with the aid of a sketch why the element of area in plane polar coordinates is $r \, dr \, d\theta$. **[3 marks]**

By using plane polar coordinates, work out the numerical value of the area integral

$$I = \int \frac{x^2 y^2}{x^2 + y^2} \, dA$$

over the interior of the circle of radius 2 centred at the origin. **[4 marks]**

2(c) Explain with the aid of a sketch how the position of a point in 3-dimensional space is specified using spherical polar coordinates (r, θ, ϕ). **[2 marks]**

Express the Cartesian coordinates (x, y, z) of a point in terms of (r, θ, ϕ). **[3 marks]**

Using the fact that the volume element in spherical polar coordinates is $dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$, evaluate the volume integral:

$$I = \int z^2 \, dv$$

over the interior of the sphere of unit radius centred at the origin. **[3 marks]**

3(a) Explain what is meant by the gradient vector grad $V \equiv \nabla V$ of a scalar field $V(x, y, z)$ in three dimensions, and give a formula expressing ∇V in terms of the partial derivatives of $V(x, y, z)$. **[3 marks]**

If a displacement is made from the position $\mathbf{r} = (x, y, z)$ to the neighbouring point $\mathbf{r} + \delta\mathbf{r} = (x + \delta x, y + \delta y, z + \delta z)$, the value of the field V changes by the amount $\delta V = V(\mathbf{r} + \delta\mathbf{r}) - V(\mathbf{r})$. Write an approximate formula for δV in terms of the gradient vector ∇V at position $\mathbf{r}$ and the displacement vector $\delta\mathbf{r}$, this formula being valid when the magnitude $|\delta\mathbf{r}|$ is very small. **[3 marks]**

Hence, explain why ∇V at any point is perpendicular to the isovalue surface of V at that point. **[3 marks]**

3(b) For the scalar field $V(\mathbf{r})$ given by $V(x, y, z) = 2x - 3y + xyz + \sin z$, calculate the numerical values of the Cartesian components of ∇V at the origin. **[2 marks]**

The equation $\mathbf{r} = (3\mathbf{i} + c\mathbf{j})t$ is the parametric equation for a straight line passing through the origin. (Here, c is a constant, and t is the variable parameter.) For what value of c is this straight line perpendicular to the gradient vector ∇V at the origin? **[4 marks]**

3(c) For the vector field $\mathbf{F}(\mathbf{r})$ given by:

$$\mathbf{F}(\mathbf{r}) = (2 + yz)\mathbf{i} + (-3 + xz)\mathbf{j} + (xy + \cos z)\mathbf{k} \ ,$$

we require to evaluate the line integral

$$I = \int \mathbf{F} \cdot d\mathbf{r}$$

for a path starting at the initial point $\mathbf{r}_0 = (0, 0, 0)$ and ending at the final point $\mathbf{r}_1 = (1, 1, 0)$. By noting the relationship between the vector field $\mathbf{F}(\mathbf{r})$ and the scalar field $V(\mathbf{r})$ given in part (b) of the present question, explain why the value of I does not depend on the path connecting the given initial and final points. **[3 marks]**

What is the numerical value of I? **[2 marks]**

4(a) A vector field $\mathbf{F}$ has the same magnitude $|\mathbf{F}| = 2$ at every point in space, and also has the same direction at every point in space. What is the flux of $\mathbf{F}$ through the circle of unit radius lying in the x-y plane and centred at the origin, in the cases where (a) $\mathbf{F}$ is parallel to the z-axis, and (b) $\mathbf{F}$ makes an angle θ with the z-axis? **[2 marks]**

The magnitude and direction of the vector field $\mathbf{F}(\mathbf{r}) = x\mathbf{i} + y\mathbf{j} + (2 + z^2)\mathbf{k}$ both depend on position $\mathbf{r}$. What is the flux of this field through the same circle? **[3 marks]**

4(b) Define the divergence $\mathrm{div}\,\mathbf{F}$ of a vector field $\mathbf{F}(\mathbf{r})$ in terms of the position-dependent Cartesian components of the field. **[2 marks]**

Write a formula expressing the "divergence theorem", i.e. the relationship between the flux of a vector field through a closed surface and the divergence of that field in the region within the surface, explaining clearly the notation that you use. **[3 marks]**

Obtain a formula for the divergence of the field $\mathbf{F}(\mathbf{r}) = x\mathbf{i} + y\mathbf{j} + (2 + z^2)\mathbf{k}$ as a function of position $\mathbf{r}$. **[2 marks]**

By using the divergence theorem, calculate the numerical value of the outward flux of $\mathbf{F}$ through the surface of the sphere of unit radius centred at the origin. **[3 marks]**

4(c) In electrostatics, the electrostatic potential field $V(\mathbf{r})$ and the electric charge density $\rho(\mathbf{r})$ are related by the equation

$$\nabla^2 V = -\rho/\epsilon_0 \,,$$

where ϵ_0 is a constant. A sphere of radius R centred at the origin has electric charge distributed within it, and the potential at any point $\mathbf{r} = (x, y, z)$ within the sphere is given by

$$V(\mathbf{r}) = 1 + r^2 \,,$$

where $r^2 = x^2 + y^2 + z^2$. Obtain a formula for the total charge inside the sphere in terms of ϵ_0 and R. **[5 marks]**

5. State Einstein's first postulate of special relativity, and explain in what way its predictions differ from those of the ether theory of light propagation. **[5 marks]**

Einstein's first postulate leads to the Lorentz equations (see Formula Sheet at the end of this examination paper). Use these equations to derive two formulas: first, the formula $L = L_0\sqrt{1 - v^2/c^2}$, relating the observed length L of an object of rest-length L_0 when it moves with speed v; and second, the formula $T = T_0 \big/ \sqrt{1 - v^2/c^2}$, relating the observed interval T between two events to the interval T_0 measured by an inertial observer for whom the two events occur at the same spatial point. **[6 marks]**

Observer A standing beside a railway track is passed by a train of rest-length 100 m travelling at speed $v = 0.5c$, where $c = 3 \times 10^8$ m s^{-1} is the speed of light. Observer B is travelling on the train, and is sitting exactly at the centre of the train. At the instant when A is passed by B, A observes the front and back of the train simultaneously struck by lightning, and the light pulses from the two lightning strikes reach A simultaneously. How long do the light pulses take to travel from the front and back of the train to A? **[4 marks]**

Light pulses from the lightning strikes do not reach B simultaneously. What is the time interval between the arrival of the light pulses at B, as measured by B? **[5 marks]**

6. State formulas for the mass m, the linear momentum $\mathbf{p}$ and the total energy E of a particle with rest mass m_0 travelling with velocity $\mathbf{u}$ relative to a given inertial frame. **[4 marks]**

Use the formulas you have stated to deduce that a particle of zero rest-mass can have energy only if it travels at the speed of light, and that its energy and momentum are related by $E = |\mathbf{p}|c$. **[3 marks]**

Hence, show that if the energy of a photon of frequency f is $E = hf$ (h is Planck's constant), then the magnitude if its momentum must be given by $|\mathbf{p}| = h/\lambda$, where λ is its wavelength. **[3 marks]**

A photon of wavelength $\lambda_0 = 5 \times 10^{-12}$ m travelling in the direction of the positive x-axis collides with a stationary electron, and is scattered back along the negative x-axis. After the collision, the electron moves with speed v along the positive x-axis. Write down equations expressing the conservation of momentum and energy in this collision, in terms of v and the wavelength λ_1 of the scattered photon. **[4 marks]**

By solving these equations, determine the numerical value of the ratio v/c. (Assume the values $c = 3 \times 10^8$ m s^{-1}, $h = 6.63 \times 10^{-34}$ J s, electron rest-mass $m_e = 9.11 \times 10^{-31}$ kg.) **[6 marks]**

7(a) Five coins are tossed simultaneously. A numerical value is given to the outcome by counting +1 for a head and −1 for a tail and adding the values. What is the probability that the numerical value is 1? **[5 marks]**

7(b) A person shoots bullets at a circular target whose radius is 15 cm. Unfortunately, the person's aim is so bad that the bullets that happen to hit the target do so completely at random, with an equal chance of hitting any point on the target. If r denotes the distance from the centre of the target of the point where a bullet hits the target, explain why the probability distribution of r is given by $P(r)\,dr = Ar\,dr$. **[3 marks]**

Determine the numerical value of the normalisation constant A. **[2 marks]**

Find also the mean value and standard deviation of r. **[4 marks]**

7(c) Students took a four-part exam question, and the probability distribution for answering n parts correctly ($n = 0, 1, 2, 3, 4$) followed the law

$$p_n = \frac{1}{8} + Cn(5-n)\,.$$

Evaluate the constant C needed to ensure that the sum of p_n over all allowed values of n is unity. **[3 marks]**

Work out the mean values $\langle n\rangle$ and $\langle n^2\rangle$ of n and n^2, and hence the standard deviation of n. **[3 marks]**

END OF PAPER
FORMULA SHEET ATTACHED

Formula Sheet

The Lorentz transformation gives the relationship between the position (x, y, z) and time t of an event in an inertial frame S and the position (x', y', z') and time t' of the same event in inertial frame S'. The frame S' moves with speed v in the positive x-direction relative to frame S, the x'- y'- and z'-axes in frame S' are parallel to the x-, y- and z-axes in frame S, and the origins of the two coordinate systems coincide at $t = t' = 0$. The equations expressing the Lorentz transformation are:

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{xv}{c^2}\right) , \end{aligned}$$

where c is the speed of light, and γ is defined as:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} .$$

Equivalently, x, y, z and t can be expressed in terms of x', y', z' and t':

$$\begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{x'v}{c^2}\right) . \end{aligned}$$