UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Physics 1B46: Mathematical Methods II

COURSE CODE	:	**PHYS1B46**
UNIT VALUE	:	**0.50**
DATE	:	**17-MAY-04**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours 30 Minutes**

04-C1076-3-160

TURN OVER

Answer **FIVE** *questions.*

A collection of useful formulas is attached after the end of the paper.

Numbers in square brackets show the provisional allocation of marks per question section.

1(a) Calculate the numerical value of the area integral:

$$I = \int_0^1 dx \int_0^{1-x} dy \, (x^2 + y^2) \; .$$

[5 marks]

1(b) With the aid of a sketch, explain how the position of a point in three-dimensional space can be specified using spherical polar coordinates r, θ, ϕ. **[2 marks]**
Give expressions for the Cartesian coordinates x, y and z in terms of r, θ and ϕ. **[2 marks]**

Use spherical polar coordinates to calculate the value of the volume integral:

$$I = \int (x^2 + y^2) \, dv$$

over the interior of the sphere of unit radius centred at the origin. **[5 marks]**

1(c) Calculate the numerical value of the following line integral in the x-y plane:

$$I = \int_C \mathbf{F} \cdot d\mathbf{s} \; ,$$

where the Cartesian components of the vector field $\mathbf{F}(\mathbf{r})$ are $F_x = 2x - y$ and $F_y = 1 - x$. The path C is the straight line starting at the point $(-1, -1)$ and ending at the point $(1, 1)$. **[4 marks]**
By applying a standard test, determine whether the value of the line integral depends on the path followed between the given initial and final points. **[2 marks]**

2(a) Obtain equations for the x- and y-components of the gradient vector ∇f of the scalar field $f(x,y) = x^2 + 2y^2 - 3x - 6y$. **[2 marks]**

Calculate the numerical value of the projection of ∇f in the direction of the unit vector $\hat{\mathbf{n}} \equiv (\mathbf{i}+\mathbf{j})/\sqrt{2}$ at the points $\mathbf{r}_0 = (0,0)$ and $\mathbf{r}_1 = (2,2)$. **[2 marks]**

Hence, deduce the point on the straight line joining the points $\mathbf{r}_0$ and $\mathbf{r}_1$ at which $f(x,y)$ has its minimum value. **[4 marks]**

2(b) Define the divergence $\operatorname{div}\mathbf{E}$ of a vector field $\mathbf{E}$ in terms of the partial derivatives with respect to position of the Cartesian components of $\mathbf{E}$. **[2 marks]**

In electrostatics, the electric field $\mathbf{E}(\mathbf{r})$ is related to the charge density $\rho(\mathbf{r})$ by the equation $\operatorname{div}\mathbf{E} = \rho/\epsilon_0$, where ϵ_0 is a constant. For a certain charge density $\rho(\mathbf{r})$, which is non-zero only inside the sphere of radius $R = 1$ centred on the origin, the electric field is given by:

$$\mathbf{E} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})r^2 \; ,$$

for $r \leq 1$, where $r = \sqrt{x^2+y^2+z^2}$ is the distance from the orgin. Calculate the charge density as a function of position inside the sphere. **[5 marks]**

2(c) A vector field $\mathbf{F}(\mathbf{r})$ is parallel to the z-axis, its x- and y-components being zero. Its z-component is given by $F_z = r^2 - r^4$, where $r = \sqrt{x^2+y^2}$ is the distance from the z-axis (*not* the distance from the origin). Obtain formulas for the three Cartesian components of $\operatorname{curl}\mathbf{F}$. **[5 marks]**

3(a) Find the general solution $y = f(x)$ of the differential equation

$$\frac{dy}{dx} = y \cos x .$$

Determine the solution satisfying the boundary condition $y = \frac{1}{2}$ at $x = 0$. **[5 marks]**

3(b) Use the "integrating factor" method to find the general solution of the differential equation

$$\frac{dy}{dx} + 2xy = e^{-x^2} .$$

[7 marks]

3(c) Determine the coefficients α and β such that $y = \alpha + \beta x$ is a particular integral of the second-order differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 2x .$$

[3 marks]

Find also the most general complementary function for this equation, noting that the complementary function should contain two independent arbitrary coefficients. **[3 marks]**

Hence, find the complete solution such that $y = 1$ and $dy/dx = 0$ at $x = 0$. **[2 marks]**

4. Consider solutions of the second-order ordinary differential equation

$$x\frac{d^2y}{dx^2} + (q+x)\frac{dy}{dx} - \alpha y = 0 ,$$

where q is a positive integer and α is a real number. If the general solution is represented as a power series:

$$y = \sum_{n=0}^{\infty} a_n x^{n+k} ,$$

with $a_0 \neq 0$, show that the only possible values of k are $k = 0$ and $k = 1 - q$. **[5 marks]**
Show also that when $k = 0$ the recurrence relation is:

$$a_{n+1}/a_n = \frac{\alpha - n}{(n+q)(n+1)} .$$

[5 marks]

Demonstrate that if α is a positive integer m, then the solution $y(x)$ is a polynomial, and determine the highest power of x that occurs in the polynomial.

[2 marks]

Write down the explicit form of the polynomial solution for the case where $m = 3$, $q = 3$ and $a_0 = 1$. **[4 marks]**
Obtain a general formula for the coefficients of the polynomial solution when m is a general positive integer and $a_0 = 1$. **[4 marks]**

5(a) Two normal 6-sided dice are tossed simultaneously. Let x be a random variable whose value is 1 if the sum of the two numbers on the dice is exactly divisible by 3, and is 0 otherwise. What is the mean value of x? **[5 marks]**

5(b) In a printed book having 300 pages, misprints occur at random, the total number of misprints in the book being 110. The probability p_n of finding n misprints on any given page is given by the formula

$$p_n = A\mu^n/n! \qquad (n = 0, 1, 2, \ldots) ,$$

where μ is the mean number of misprints on each page. Use the mean number of misprints on each page to calculate the numerical value of the normalisation constant A. [If you wish, you may assume that the probability of finding more than 4 misprints on any page is negligibly small, so that p_n can be approximated by 0 for $n > 4$.] **[4 marks]**

Calculate also the standard deviation of the number of misprints on a page. **[4 marks]**

5(c) For the hydrogen atom in its ground state, the probability distribution $P(r)\,dr$ for finding the electron a distance between r and $r + dr$ from the proton is given by

$$P(r)\,dr = Ar^2 e^{-2r/a_0}\,dr ,$$

where $a_0 = 0.529$ Å is the Bohr radius. Here, r can have any value between 0 and ∞. Show that the normalisation constant A is given by the formula $A = 4/a_0^3$. **[3 marks]**

Calculate also the mean value of r and its standard deviation, giving your answers in Å units, correct to 10^{-3} Å. **[4 marks]**

[You may use the general formula:

$$\int_0^\infty t^n e^{-\alpha t}\,dt = \frac{n!}{\alpha^{n+1}} \;]$$

6(a) State Einstein's hypotheses of special relativity. Explain briefly how the hypotheses lead to the conclusion that events that are simultaneous in one inertial frame may not be simultaneous in another. **[4 marks]**

A train having a rest-length of 100 m travels at a speed of $0.4c$, where c is the speed of light. What is the length of the train as measured by an observer standing on the ground? (You may assume the standard formula for the length of a moving object, or derive it from the Lorentz transformations given in the Formula Sheet.) **[3 marks]**

The train enters a tunnel, whose length is also 100 m. As seen by the observer on the ground, for what period of time is the entire length of the train inside the tunnel? (Assume the numerical value $c = 3 \times 10^8$ m s^{-1}.) **[2 marks]**

The train-driver sees the tunnel as being shorter than his train, and claims that the train is *never* entirely inside the tunnel. Explain qualitatively how the observer on the ground and the train-driver can both be correct about the train's being (or not being) entirely inside the tunnel. **[3 marks]**

6(b) Use the Lorentz transformations to derive the formula for the Doppler shifted frequency f of a light-source as seen by an observer travelling at speed v directly towards the light-source, whose rest-frequency is f_0:

$$f = f_0 \sqrt{\frac{1 + v/c}{1 - v/c}} \,.$$

[5 marks]

A motorist passes a red traffic-light (wavelength = 650 nm), and is stopped by the police. The motorist tells the officer that he observed the traffic-light as being green (wavelength = 500 nm). What was the motorist's speed? (Assume that the motorist was travelling directly towards the traffic-light.) **[3 marks]**

7. State formulas for the mass m, the linear momentum $\mathbf{p}$ and the total energy E of a particle with rest-mass m_0 travelling with velocity $\mathbf{u}$ relative to a frame of reference S. Hence, prove that the quantity $c^2|\mathbf{p}|^2 - E^2$ is invariant (i.e. independent of the velocity $\mathbf{u}$), and determine its value. **[5 marks]**

A particle with rest-mass M_0 spontaneously decays at rest into two particles with rest-masses m_1 and m_2, which emerge along the positive and negative x-axes with linear momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, the corresponding total energies being E_1 and E_2. Using the principle of conservation of linear momentum, and the relationship between linear momentum and total energy for each particle, show that the quantity $E_1^2 - E_2^2$ is a constant depending only on the rest-masses m_1 and m_2 of the two particles produced in the decay process. **[5 marks]**

For the case where the initial particle is a pion ($M_0 = 135$ MeV/c^2), and the two particles into which the pion decays are a muon ($m_1 = 103$ MeV/c^2) and an antineutrino (assume $m_2 = 0$), use the principle of energy conservation to calculate the kinetic energies (eV units) of the muon and the antineutrino in the rest-frame of the pion. **[10 marks]**

END OF PAPER
FORMULA SHEETS ATTACHED

1B46 Formula Sheet

Calculus

(All angles are in radians)

$$\frac{d}{dx}\sin x = \cos x\,, \qquad \frac{d}{dx}\cos x = -\sin x\,,$$

$$\frac{d}{dx}\sinh x = \cosh x\,, \qquad \frac{d}{dx}\cosh x = \sinh x\,,$$

$$\frac{d}{dx}\ln x = \frac{1}{x}\,, \qquad \frac{d}{dx}e^x = e^x\,,$$

Series

$$\sum_{n=0}^{N} n = \frac{1}{2}N(N+1)\,, \qquad \sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}\,,$$

$$(1+x)^n = 1 + n\,x + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots + \binom{n}{m}x^m + \ldots$$

If n is a positive integer then this is a finite series. Otherwise it is an infinite series which only converges for $|x| < 1$.

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty}\frac{x^n}{n!}\,,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n+1}\frac{x^n}{n} + \ldots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \quad (-1 < x \leq +1),$$

Vectors

If $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$, and similarly for the vector $\mathbf{B}$, then the scalar (dot) product is given by

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z = A\,B\,\cos\theta\,,$$

where A and B are the magnitudes of the two vectors and θ the angle between them.

The vector (cross) product can be expressed as a three-by-three determinant

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= \mathbf{i}\,(A_yB_z - A_zB_y) - \mathbf{j}\,(A_xB_z - A_zB_x) + \mathbf{k}\,(A_xB_y - A_yB_x) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}\,.\end{aligned}$$

The magnitude of the vector product is given by

$$|\mathbf{A}\times\mathbf{B}| = AB\,|\sin\theta|\,.$$

Relativity

The Lorentz transformations give the relationship between the position (x, y, z) and time t of an event in an inertial frame S and the position (x', y', z') and time t' of the same event in inertial frame S'. The frame S' moves with speed v in the positive x-direction relative to frame S, the x'-, y'- and z'-axes in frame S' are parallel to the x-, y- and z-axes in frame S, and the origins of the two coordinate systems coincide at $t = t' = 0$. The transformations are:

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{xv}{c^2}\right)\,, \end{aligned}$$

where c is the speed of light, and γ is defined as:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\,.$$