Physical Dynamics (SPA5304) – Exercise Class
Week 8 (10-Mar-2017)

Problem 1

Consider a cube of homogeneous density, mass m and side a. Choose the origin O' of the body-fixed system to be at the one corner and the three edges adjacent to that corner to lie in the positive $\hat{x}$, $\hat{y}$, $\hat{z}$ axes.

Show, by performing a straightforward three-dimensional integration of the expression

$$I^{ab} = \int d^3x\, \rho(\mathbf{x}) \left(\delta^{ab}\mathbf{x}^2 - \mathbf{x}^a\mathbf{x}^b\right),$$

(where in our case the density ρ is constant and can therefore be pulled out of the integral) that the inertia tensor about the origin O' of the fixed body system is given by

$$I_{O'} = \begin{pmatrix} \frac{2}{3}b & -\frac{1}{4}b & -\frac{1}{4}b \\ -\frac{1}{4}b & \frac{2}{3}b & -\frac{1}{4}b \\ -\frac{1}{4}b & -\frac{1}{4}b & \frac{2}{3}b \end{pmatrix}$$

where $b := Ma^2$.

Hint: In principle I, being a symmetric 3×3 matrix, has only 6 independent elements. Using the symmetries of the cube, argue that in fact in this specific case there only two independent quantities to calculate.

Problem 2

Consider a homogeneous cube suspended by one of its edges. The mass of the cube is m and the length of its edges is ℓ. Gravity acts vertically. Find the frequency of small oscillations.

Problem 3

A rigid rod of length $2L$ has its lower end in contact with a smooth horizontal plane. The mass of the rod is equal to m, and its mass density is uniform. Gravity acts as usual in the vertical direction. The rod is initially at an angle α with respect to the vertical, when it is released from rest. The motion is constrained to take place in the vertical plane $(\hat{x}, \hat{y})$.

i) How many degrees of freedom does the system have?

   Hint: The y coordinate Y of the center of mass G and the angle formed by the rod and the vertical are not independent degrees of freedom. I suggest picking the latter as one of the generalized coordinates.

ii) Choosing appropriate generalized coordinates, write down the Lagrangian of the system.

Hint: To answer this question you will also need the moment of inertia of the rod with respect to an axis orthogonal to the rod and passing through its center of mass G. This has been calculated in the lecture, so take the result from there.

iii) Show that in the subsequent motion after the rod is released from rest, the x coordinate X of the center of mass G remains constant.

iv) Write down the energy of the system. Use energy conservation to determine how long it will take the rod to hit the ground after it is released from rest, if the initial angle α with respect to the vertical is equal to $\pi/3$.

Hint: Use energy conservation to determine $\dot{\theta}^2$, where θ is the angle between the rod and the vertical. From this, write down $dt/d\theta = 1/\dot{\theta}$. You will then need the following integral:

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta \left(\frac{1 + 3\sin^2\theta}{1 - 2\cos\theta} \right)^{\frac{1}{2}} \approx 2.04\,.$$