Physical Dynamics (SPA5304) – Exercise Class
Week 7 (3-Mar-2017)

Problem 1

A rigid rod of mass M and homogeneous density is free to move on the plane. Determine the Lagrangian and the derive the Euler-Lagrange equations.

Problem 2

A particle moves freely in the gravitational field of a fixed mass distribution. Find the conservation principles that correspond to the symmetries of the following fixed mass distributions:

i) a uniform sphere,

ii) a uniform half plane,

iii) two particles,

iv) a uniform right circular cone,

v) an infinite uniform circular cylinder.

Problem 3

Consider the motion of a particle of mass m described by the following Lagrangian:

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + k\left(x\dot{y} - y\dot{x}\right).$$

[Such a Lagrangian could be used to describe the motion of a charged particle in a constant magnetic field directed along the $\hat{z}$ axis; however, you don't need to know this in order to solve the problem.]

i) Write the Euler-Lagrange equations, and use them to show that the three quantities

$$\mathcal{I}_x := m\dot{x} - 2ky\,, \qquad \mathcal{I}_y := m\dot{y} + 2kx\,, \qquad \mathcal{I}_z := m\dot{z}\,,$$

are separately conserved.

ii) Show that the Lagrangian L is exactly invariant (i.e. $\delta L = 0$) under a translation along the $\hat{z}$ axis, and invariant under translations along the $\hat{x}$ and $\hat{y}$ axes up to a total time derivative of a function. Show that the corresponding Noether invariants are given by $(\mathcal{I}_x,\, \mathcal{I}_y,\, \mathcal{I}_z)$ defined above.

iii) Show that L is invariant under a rotation by an infinitesimal angle ϵ about the $\hat{z}$ axis,

$$\delta x = \epsilon y\,, \qquad \delta y = -\epsilon x\,,$$

and show that the corresponding Noether invariant is given by

$$\tilde{\mathcal{I}} := (x\dot{y} - y\dot{x}) + k\left(x^2 + y^2\right).$$

iv) Use the expresions on $\mathcal{I}_x$, $\mathcal{I}_y$ given in question (i) above to eliminate $\dot{x}$ and $\dot{y}$ from the expression of $\tilde{\mathcal{I}}$ derived in (iii), thus showing that the trajectory of the particle lies on the cylinder of equation

$$\left(x - \frac{\mathcal{I}_y}{2k}\right)^2 + \left(y + \frac{\mathcal{I}_x}{2k}\right)^2 = \frac{\mathcal{I}_x^2 + \mathcal{I}_y^2}{4k^2} - \frac{\tilde{\mathcal{I}}}{k}\,.$$

v) Use the equations of question (i) to show that the trajectory is a helix.