University College London
Department of Physics and Astronomy
2246E Mathematical Methods III
Coursework M4 (2007-2008)

Solutions to be handed in on Wednesday, January, 9th, 2007

1. (a) Show that if for two matrices $\underline{A}$ and $\underline{B}$ the product $\underline{AB}$ is defined that

$$(\underline{AB})^T = \underline{B}^T \underline{A}^T$$ **[2 mark]**

(b) Given are the matrices

$$\underline{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{pmatrix} \qquad \underline{B} = \begin{pmatrix} -1 & 0 \\ 4 & 2 \\ 3 & -1 \end{pmatrix} \qquad \underline{C} = \begin{pmatrix} 7 & -1 & 2 \end{pmatrix}$$

Give $\underline{A}^T$, $\underline{B}^T$ and $\underline{C}^T$. **[3 marks]**

(c) Which of the following matrix products are possible? Evaluate the ones which are possible:
$\underline{AB}$, $\underline{AB}^T$, $\underline{BA}$, $\underline{B}^T\underline{A}^T$, $\underline{AC}$, $\underline{CA}$, $\underline{CB}$, $\underline{B}^T\underline{C}^T$, $\underline{CC}$. **[9 marks]**

(d) Calculate $\underline{A}^{-1}$. **[6 marks]**

CONTINUED

2. The real quadratic form F in three dimensions is given by:

$$F = 2x^2 - 8xy + 2y^2 + 4z^2 = 0 \, ,$$

(a) Write down the matrix $\underline{A}$ so that F is given by

$$F = \underline{v}^T \underline{A}\underline{v} = 0 \, ,$$

with $\underline{v}^T = \begin{pmatrix} x & y & z \end{pmatrix}$. **[2 marks]**

(b) Find the three different eigenvalues of $\underline{A}$ by writing the characteristic equation in the form

$$(p - \lambda) \left\{ (q - \lambda)^2 - r \right\} = 0$$

and calculating the values of p, q and r. Calculate the three corresponding normalized eigenvectors. **[12 marks]**

(c) Evaluate the transformation matrix $\underline{S}$, for which $\underline{S}^T \underline{A}\underline{S}$ is diagonal. **[2 marks]**

Set $\underline{u} = \underline{S}^T \underline{v}$ and write the quadratic form F in the new variables $\underline{u}^T = \begin{pmatrix} \tilde{x} & \tilde{y} & \tilde{z} \end{pmatrix}$. **[4 marks]**

CONTINUED

3. Solve

$$2(x^2 + x^3)\frac{d^2y}{dx^2} - (x - 3x^2)\frac{dy}{dx} + y = 0 \ ,$$

with a general series solution. Write the differential equation in the general form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \ .$$

Evaluate the singular points of the differential equation. The equation has a series solution of the form **[2 marks]**

$$y = \sum_{n=0}^{\infty} a_n x^{n+k} \ .$$

Write down the indicial equation and show that $k = \frac{1}{2}$ and $k = 1$. **[3 marks]**
Show that the recursion relations are given by

$$a_{n+1} = -a_n \ .$$ **[8 marks]**

Give the radius of convergence of these series. Calculate the first 4 terms of the series solution and show that the general solution can be written in the form **[2 marks]**

$$y(x) = \frac{Ax + B\sqrt{x}}{1 + x} \ .$$ **[5 marks]**

CONTINUED

4. The function $f(x)$ is defined on the intervall $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$ with

$$f(x) = 2\cos x$$

(a) Is $f(x)$ and even or odd function? **[1 mark]**

(b) The Fourier expansion is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\ ,$$

with $-L \leq x \leq L$. Show that the Fourier coefficients of $f(x) = 2\cos x$ with $L = \frac{\pi}{4}$ are given by

$$\begin{aligned} a_n &= \frac{4}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(x)\cos(4nx)\ dx \\ b_n &= 0\ . \end{aligned}$$

[7 marks]

(c) Evaluate the coefficients a_n and show

$$f(x) = \frac{4\sqrt{2}}{\pi} + \frac{8\sqrt{2}}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{1}{1-16n^2}\cos 4nx\ .$$

[8 marks]

Hints:
$\int \cos ax \cos bx\ dx = \frac{1}{2}\left(\frac{\sin(a-b)x}{a-b} + \frac{\sin(a+b)x}{a+b}\right)$
and
$\sin(a+b) = \sin a \cos b + \cos a \sin b$.

(d) By considering $f(x)$ at $x = \pi/4$ calculate the value of the series

$$\sum_{n=1}^{\infty} \frac{1}{16n^2 - 1}$$

[4 marks]

CONTINUED

5. (a) Use

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

with $P_0(x) = 1$ and $P_1(x) = x$ to calculate $P_2(x)$ and $P_3(x)$. Then express

i. $3x^2 + x - 1$

ii. $x - x^3$

in terms of a finite series of Legendre polynomials. **[6 marks]**

(b) The generating function $g(x,t)$ is related to the Legendre polynomials via

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x) \ . \qquad (1)$$

Show

$$(x-t)\frac{\partial g}{\partial x} = t\frac{\partial g}{\partial t} \qquad (2)$$

[4 marks]

(c) By substituting the series from equation (1) into equation (2) show that

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x) \qquad (3)$$

where the prime denotes the derivative with respect to x. **[5 marks]**

(d) Differentiate

$$(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x)$$

with respect to x and eliminate P_{n-1}' with the help of equation (3). What is the resulting equation? **[5 marks]**

CONTINUED

6. The Schrödinger equation for a particle of mass m in a one dimensional potential $V(x)$ is given by

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

(a) If you write $\Psi(x,t) = F(x) \times T(t)$ show that the solution of the differential equation is of the form

$$T(t) = Ce^{-iEt/\hbar}$$ **[3 marks]**

(b) Show, that for zero potential ($V(x) \equiv 0$), the solution is given by

$$\Psi(x,t) = \{A\cos kx + B\sin kx\}\, e^{-iEt/\hbar} \qquad (4)$$ **[2 marks]**

Further show that k and E are related by

$$k^2 = \frac{2m}{\hbar^2}E$$ **[1 mark]**

.

(c) Assume now that $V = 0$ for $0 \leq x \leq l$ and $\Psi(x,t) = 0$ at $x = 0$ and $x = l$ for all times t. Show that the general solution fullfilling these boundary conditions is

$$\Psi(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-iE_n t/\hbar}$$

Give E_n as a function of n. **[8 marks]**

(d) $|\Psi|^2$ is the probability of finding a particle at position x. Show that in order to to ensure

$$\int_0^l |\Psi|^2\, dx = 1\ .$$

the coefficients B_n have to obey

$$\sum_{n=0}^{\infty} |B_n|^2 = \frac{2}{l}$$ **[6 marks]**

Hint: $\int_{-\pi}^{\pi} \sin mx \sin nx\; dx = \pi\delta_{nm}$.

HAPPY HOLIDAYS !!!!!

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Model Answers

1. (a) Define $\underline{C} = \underline{AB}$, hence

$$c_{ij} = \sum_k a_{ik} b_{kj}$$

and then

$$\left(C^T)\right)_{ji} = c_{ij} = \sum_k a_{ik} b_{kj} = \sum_k \left(A^T\right)_{ki} \left(B^T\right)_{jk} = \sum_k \left(B^T\right)_{jk} \left(A^T\right)_{ki} = \left(B^T A^T\right)_{ji}$$

hence the expression given is proven.

Note that this is entirely book work.

(b)

$$\underline{A}^T = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\underline{B}^T = \begin{pmatrix} -1 & 4 & 3 \\ 0 & 2 & -1 \end{pmatrix}$$

$$\underline{C}^T = \begin{pmatrix} 7 \\ -1 \\ 2 \end{pmatrix}$$

(c) $\underline{A} \in M_{3,3}$, $\underline{B} \in M_{3,2}$, $\underline{C} \in M_{1,3}$, $\underline{A}^T \in M_{3,3}$, $\underline{B}^T \in M_{2,3}$, $\underline{C}^T \in M_{3,1}$.

$$\underline{AB} = \begin{pmatrix} 6 & 1 \\ 11 & 3 \\ 4 & -2 \end{pmatrix}$$

$\underline{AB}^T$ not possible.

$\underline{BA}$ not possible.

$$\underline{B}^T \underline{A}^T = \begin{pmatrix} 6 & 11 & 4 \\ 1 & 3 & -2 \end{pmatrix}$$

$\underline{AC}$ not possible.

$$\underline{CA} = \begin{pmatrix} 11 & 5 & 10 \end{pmatrix}$$

$$\underline{CB} = \begin{pmatrix} -5 & -4 \end{pmatrix}$$

$$\underline{B}^T \underline{C}^T = \begin{pmatrix} -5 \\ -4 \end{pmatrix}$$

$\underline{CC}$ not possible.

(d)

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 \end{array}\right)$$

Multiply first row by two and subtract from last row

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 1 \end{array}\right)$$

Add last row to second row

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \\ 0 & -2 & 0 & -2 & 0 & 1 \end{array}\right)$$

Divide last row by -2 and exchange 2nd and 3rd row

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right)$$

Subtract 2nd and 3rd row from 1st

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1/2 \\ 0 & 1 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right)$$

$$\underline{A}^{-1} = \begin{pmatrix} 2 & -1 & -1/2 \\ 1 & 0 & -1/2 \\ -2 & 1 & 1 \end{pmatrix}$$

Of course the students can also solve this by using adjoints.

2. (a)

$$\underline{A} = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

(b)

$$|\underline{A} - \lambda \underline{I}| = 0$$

$$\Leftrightarrow \begin{vmatrix} 2-\lambda & -4 & 0 \\ -4 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix}$$

$$= (4-\lambda) \begin{vmatrix} 2-\lambda & -4 \\ -4 & 2-\lambda \end{vmatrix}$$

$$= (4-\lambda)\left\{(2-\lambda)^2 - 16\right\} = 0$$

$$\lambda_1 = 4 \qquad \vee \qquad \begin{aligned} &(2-\lambda)^2 = 16 \\ &2-\lambda = \pm 4 \\ &\lambda_{2,3} = 2 \pm 4 \\ &\Rightarrow \lambda_2 = 6 \\ &\quad \lambda_3 = -2 \end{aligned}$$

Eigenvectors for $\lambda_1 = 4$:

$$\begin{pmatrix} -2 & -4 & 0 \\ -4 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Leftrightarrow$

$$-2x - 4y = 0$$

$$-4x - 2y = 0$$

$\Rightarrow$

$$-2y = 4x$$

$\Rightarrow$

$$y = -2x$$

$\Rightarrow$

$$-2x + 8x = 0$$

$\Rightarrow$

$$x = 0 \qquad y = 0 \qquad z \text{ arbitrary}$$

$\Rightarrow$

$$\underline{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Eigenvector for $\lambda_2 = 6$:

$$\begin{pmatrix} -4 & -4 & 0 \\ -4 & -4 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$

$$\begin{aligned} 4x + 4y &= 0 \\ 2z &= 0 \end{aligned}$$

$\Rightarrow \qquad z = 0$ and
$y = -x$
$\Rightarrow$

$$\underline{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Eigenvector for $\lambda_3 = -2$:

$$\begin{pmatrix} 4 & -4 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \qquad z = 0$
$\Rightarrow$

$$4x - 4y = 0$$

$\Rightarrow \qquad x = y$
$\Rightarrow$

$$\underline{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

(c)

$$\underline{S} = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$

$$\underline{D} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

(d)

$$F = \underline{u}^T \underline{D u} = 0$$

$\Rightarrow$

$$\begin{pmatrix} \tilde{x} & \tilde{y} & \tilde{z} \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = 0$$

$$4\tilde{x}^2 + 6\tilde{y}^2 - 2\tilde{z}^2 = 0$$

3.

$$p(x) = -\frac{x - 3x^2}{2(x^2 + x^3)} \; ; \qquad q(x) = \frac{1}{2(x^2 + x^3)}$$

Singular points

$$x^2 + x^3 = 0 \quad \Leftrightarrow \quad x^2(1 + x) = 0 \quad \Leftrightarrow \quad x_{1,2} = 0 \quad \wedge \quad x_3 = -1$$

$$p_0 = \lim_{x\to 0} xp(x) = -\lim_{x\to 0}\left[\frac{x^2 - 3x^3}{2(x^2 + x^3)}\right] = -\lim_{x\to 0}\frac{1 - 3x}{2(1 + x)} = -\frac{1}{2}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0}\frac{x^2}{2(x^2 + x^3)} = \lim_{x\to 0}\frac{1}{2(1 + x)} = \frac{1}{2}$$

indicial equation

$$k(k - 1) + p_0 k + q_0 = k(k - 1) - \frac{k}{2} + \frac{1}{2} \overset{!}{=} 0$$

$\Leftrightarrow$

$$\left(k - \frac{3}{4}\right)^2 + \frac{1}{2} - \frac{9}{16} = \left(k - \frac{3}{4}\right)^2 - \frac{1}{16} \overset{!}{=} 0$$

$\Leftrightarrow$

$$k - \frac{3}{4} = \pm\frac{1}{4} \quad \Leftrightarrow \quad k_{1,2} = \frac{3}{4} \pm \frac{1}{4}$$

$$k_1 = 1 \quad ; \qquad k_2 = \frac{1}{2}$$

Use ansatz:

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^{n+k} \\ y'(x) &= \sum_{n=0}^{\infty} (n + k) a_n x^{n+k-1} \\ y''(x) &= \sum_{n=0}^{\infty} (n + k)(n + k - 1) a_n x^{n+k-2} \end{aligned}$$

$$2(x^2+x^3)\sum_{n=0}^{\infty}(n+k)(n+k-1)a_n x^{n+k-2} - (x-3x^2)\sum_{n=0}^{\infty}(n+k)a_n x^{n+k-1} + \sum_{n=0}^{\infty} a_n x^{n+k} = 0$$

$\Leftrightarrow$

$$\begin{aligned} &\sum_{n=0}^{\infty} a_n x^{n+k} \left[2(n+k)(n+k-1) - (n+k) - 1\right] \\ + &\sum_{n=0}^{\infty} a_n x^{n+k+1} \left[2(n+k)(n+k-1) + 3(n+k)\right] \quad = 0 \end{aligned}$$

In the second sum replace $n+1 \rightarrow n$

$$\begin{aligned} &\sum_{n=0}^{\infty} a_n x^{n+k} \left[2(n+k)(n+k-1) - (n+k) - 1\right] \\ + &\sum_{n=1}^{\infty} a_{n-1} x^{n+k} \left[2(n+k-1)(n+k-2) + 3(n+k-1)\right] \quad = 0 \end{aligned}$$

$n = 0$ equation is fullfilled due to indicial equation and for $n \geq 1$ we obtain

$$a_n \left[(n+k)(2n+2k-3) + 1\right] + a_{n-1} \left[(n+k-1)(2n+2k-1)\right] = 0$$

$\Leftrightarrow$

$$\left(2n^2 + 4nk - 3n + 2k^2 - 3k + 1\right) a_n + \left(2n^2 + 4kn - n + 2k^2 - k - 2n - 2k + 1\right) a_{n-1} = 0$$

$\Leftrightarrow$

$$a_n = -a_{n-1}$$

or

$$a_{n+1} = -a_n$$

Since the next singularity is reached for $x = -1$ the radius of convergence is $|x| < 1$.

Set $a_0 = 1$, then obtain general solution

$$y(x) = x^k \sum_{n=0}^{\infty} a_n x^n = x^k \left(1 - x + x^2 - x^3 + ...\right)$$

This is the progression of

$$y(x) = \frac{x^k}{1+x}$$

with $|x| < 1$.

Hence the general solution is

$$y(x) = \frac{A\sqrt{x} + Bx}{1 + x}$$

Note that we might consider giving an extra point for those who notice that the actual general solution should be

$$y(x) = \frac{A\sqrt{|x|} + Bx}{1 + x}$$

4. (a) We notice that $f(x)$ is an *even* function.

(b)

$$f(x) = 2\cos x$$

We multiply equation (4b) on both sides with $\sin(4mx)$ and integrate over x from $-\frac{pi}{4}$ to $+\frac{\pi}{4}$. Since $f(x)$ is even and $\sin(4mx)$ is odd, the left hand side is zero. And we obtain

$$0 = \int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} \left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(4nx)\right) \sin(4mx)\, dx + \int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} \left(\sum_{n=1}^{\infty} b_n \sin(4nx)\right) \sin(4mx)\, dx$$

Due to the orthogonality of the sine and cosine function we obtain

$$0 = \sum_{n=1}^{\infty} b_n \frac{\pi}{4}\delta_{mn} = \frac{\pi}{4} b_m$$

and hence $b_m = 0$. We now proceed with multiplying equation (4b) on both sides with $\cos(4mx)$ and integrate over x from $-\frac{\pi}{4}$ to $+\frac{\pi}{4}$. Now the left hand side does not vanish and we obtain

$$\int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} f(x)\cos(4mx)\, dx = \int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} \left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(4nx)\right) \cos(4mx)\, dx + \int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} \left(\sum_{n=1}^{\infty} b_n \sin(4nx)\right) \cos(4mx)\, dx$$

Now the second term vanishes and we obtain with the orthogonality relation and we obtain

$$\int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} f(x)\cos(4mx)\, dx = \sum_{n=1}^{\infty} a_n \int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} \cos(4nx)\cos(4mx)\, dx = \sum_{n=1}^{\infty} a_n \frac{\pi}{4}\delta_{mn} = a_m \frac{\pi}{4}$$

for $m \neq 0$ and

$$\int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} f(x)\, dx = \frac{a_0}{2} 2\frac{\pi}{4}$$

for $m = 0$, and hence the overall relation given.

Note that this is entirely book work with $L = \frac{\pi}{4}$

(c) Use $L = \pi/4$

$$a_n = \frac{4}{\pi}\int_{-\pi/4}^{\pi/4} 2\cos x\cos(4nx)\,dx = \frac{16}{\pi}\int_0^{\pi/4} \cos x\cos(4nx)\,dx$$

Use the given integral

$$a_n = \frac{8}{\pi}\left[\frac{\sin(1-4n)x}{1-4n} + \frac{\sin(1+4n)x}{1+4n}\right]\Bigg|_0^{\pi/4} = \frac{8}{\pi}\left\{\frac{\sin\left(\frac{\pi}{4}-n\pi\right)}{1-4n} + \frac{\sin\left(\frac{\pi}{4}+n\pi\right)}{1+4n}\right\}$$

Now we use the given trigonometrical identity and obtain

$$\sin\left(\frac{\pi}{4}-n\pi\right) = \sin\frac{\pi}{4}\cos n\pi - \cos\frac{\pi}{4}\sin n\pi = \frac{1}{2}\sqrt{2}(-1)^n$$

and

$$\sin\left(\frac{\pi}{4}+n\pi\right) = \sin\frac{\pi}{4}\cos n\pi + \cos\frac{\pi}{4}\sin n\pi = \frac{1}{2}\sqrt{2}(-1)^n$$

so we obtain

$$a_n = \frac{4}{\pi}\sqrt{2}(-1)^n\left[\frac{1}{1-4n} + \frac{1}{1+4n}\right] = \frac{8\sqrt{2}}{\pi}(-1)^n\frac{1}{1-16n^2}$$

and finally

$$a_0 = 2\frac{8}{\pi}\int_0^{\pi/4}\cos x\,dx = \frac{16}{\pi}\sin\frac{\pi}{4} = \frac{8}{\pi}\sqrt{2}$$

and we obtain

$$f(x) = \frac{4\sqrt{2}}{\pi} + \frac{8\sqrt{2}}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{1}{1-16n^2}\cos 4nx\ .$$

(d) The given function at $x = \pi/4$ is

$$f\left(\frac{\pi}{4}\right) = 2\cos\frac{\pi}{4} = \sqrt{2}$$

the Fourier expansion at this point is

$$f\left(\frac{\pi}{4}\right) = \frac{4\sqrt{2}}{\pi} + \sum_{n=1}^{\infty}\frac{8\sqrt{2}}{\pi}(-1)^n\frac{1}{1-16n^2}\underbrace{\cos(n\pi)}_{(-1)^n} \stackrel{!}{=} \sqrt{2}$$

we obtain

$$\frac{\pi}{8} - \frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{1 - 16n^2}$$

and hence

$$\frac{1}{2} - \frac{\pi}{8} = \sum_{n=1}^{\infty} \frac{1}{16n^2 - 1}$$

5. (a)

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

with

$$P_0(x) = 1 \quad ; \qquad P_1(x) = x$$

Therefore

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

and

$$3P_3(x) = 5x\left(\frac{3}{2}x^2 - \frac{1}{2}\right) - 2x = \frac{15}{2}x^3 - \frac{9}{2}x$$

$\Rightarrow$

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$

hence

$$2P_2(x) = 3x^2 - 1 = 3x^2 - P_0(x)$$

and

$$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$$

and further

$$2P_3(x) = 5x^3 - 3x$$

$\Rightarrow$

$$2P_3(x) + 3P_1(x) = 5x^3$$

$\Rightarrow$

$$x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$$

$\Rightarrow$

i.

$$3x^2+x-1 = 3\left(\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right)+P_1(x)-P_0(x) = 2P_2(x)+P_1(x)$$

ii.

$$x - x^3 = P_1(x) - \frac{2}{5}P_3(x) - \frac{3}{5}P_1(x) = \frac{2}{5}\left(P_1(x) - P_3(x)\right)$$

(b)

$$g(x,t) = \left(1 - 2xt + t^2\right)^{-1/2}$$

$$\frac{\partial g(x,t)}{\partial x} = (-2t)\left(-\frac{1}{2}\right)\frac{1}{(1-2xt+t^2)^{3/2}} = \frac{t}{(1-2xt+t^2)^{3/2}}$$

$$\frac{\partial g(x,t)}{\partial t} = (-2x+2t)\left(-\frac{1}{2}\right)\frac{1}{(1-2xt+t^2)^{3/2}} = \frac{x-t}{(1-2xt+t^2)^{3/2}}$$

$\Rightarrow$

$$(x-t)\frac{\partial g(x,t)}{\partial x} = \frac{t(x-t)}{(1-2xt+t^2)^{3/2}}$$

$$t\frac{\partial g(x,t)}{\partial t} = \frac{t(x-t)}{(1-2xt+t^2)^{3/2}}$$

$\Rightarrow$

$$(x-t)\frac{\partial g(x,t)}{\partial x} = t\frac{\partial g(x,t)}{\partial t}$$

(c)

$$(x-t)\sum_{n=0}^{\infty} t^n P_n'(x) \stackrel{!}{=} t\sum_{n=0}^{\infty} n t^{n-1} P_n(x) = \sum_{n=0}^{\infty} n t^n P_n(x)$$

$$\sum_{n=0}^{\infty} t^n \left[x P_n'(x) - n P_n(x)\right] = \sum_{n=0}^{\infty} t^{n+1} P_n'(x)$$

on the right hand side substitute $n \to n-1$

$$\sum_{n=0}^{\infty} t^n \left[x P_n'(x) - n P_n(x)\right] = \sum_{n=1}^{\infty} t^{n} P_{n-1}'(x)$$

and we obtain for $n \geq 1$

$$x P_n'(x) - n P_n(x) = P_{n-1}'(x)$$

(d)

$$\left(1 - x^2\right) P_n'(x) = n P_{n-1}(x) - n x P_n(x)$$

Differentiate wrt x

$$-2x P_n'(x) + \left(1 - x^2\right) P_n''(x) = n P_{n-1}'(x) - n P_n(x) - n x P_n'(x)$$

from equation (3):

$$P'_{n-1}(x) = xP'_n(x) - nP_n(x)$$

$\Rightarrow$

$$-2xP'_n(x)+\left(1-x^2\right)P''_n(x) = n\left[xP'_n(x) - nP_n(x)\right] - nP_n(x) - nxP'_n(x)$$

$\Leftrightarrow$

$$\left(1-x^2\right)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0$$

This is the Legendre equation.

6. (a) Seperation Ansatz

$$\Psi = F(x)T(t)$$

$$-\frac{\hbar^2}{2m}T\partial_x^2 F + VFT = i\hbar F\partial_t T$$

divide by FT

$$-\frac{\hbar^2}{2m}\frac{\partial_x^2 F}{F} + V = i\hbar\frac{\partial_t T}{T} \equiv E$$

both sides of the equation are set to E, which is constant. Temporal equation

$$i\hbar\frac{\partial_t T}{T} = E$$

$\Rightarrow$

$$T = C\exp\left(-i\frac{Et}{\hbar}\right)$$

(b) $V \equiv 0$

$\Rightarrow$

$$-\frac{\hbar^2}{2m}\frac{\partial_x^2 F}{F} = E$$

$\Rightarrow$

$$\partial_x^2 F = -\frac{2m}{\hbar^2}EF$$

$\Rightarrow$

$$F(x) = A\cos kx + B\sin kx$$

with $k^2 = \frac{2m}{\hbar^2}E$.

(c) General solution

$$F(x) = \sum_k A_k \cos kx + B_k \sin kx$$

hence $\Psi(0,t) = 0 \quad \Rightarrow$ only $\sin kx$.

$$\Psi(l,t) = \sum_k B_k \sin kl \exp\left(-iEt/\hbar\right) \stackrel{!}{=} 0$$

$\Rightarrow$

$$kl = n\pi \quad \Rightarrow \quad k = \frac{n\pi}{l}$$

$\Rightarrow$

$$\frac{n^2\pi^2}{l^2} = \frac{2mE_n}{\hbar^2}$$

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2ml^2}$$

$\Rightarrow$

$$\Psi(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{iE_n t}{\hbar}}$$

(d)

$$|\Psi|^2 = \Psi\Psi^* = \left[\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-i\frac{E_n t}{\hbar}}\right]\left[\sum_{m=1}^{\infty} B_m^* \sin\frac{m\pi x}{l} e^{i\frac{E_m t}{\hbar}}\right]$$

$\Rightarrow$

$$\int |\Psi|^2 \, dx = \sum_{n,m=1}^{\infty} B_n B_m^* e^{-i\frac{t}{\hbar}(E_n - E_m)} \int_0^l \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} \, dx$$

$$= \frac{1}{2} \sum_{n,m=1}^{\infty} B_n B_m^* e^{-i\frac{t}{\hbar}(E_n - E_m)} \int_{-l}^{l} \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} \, dx$$

Substitute $u = \frac{\pi x}{l}$

$$= \frac{l}{2\pi} \sum_{n,m=1}^{\infty} B_n B_m^* e^{-i\frac{t}{\hbar}(E_n - E_m)} \underbrace{\int_{-\pi}^{\pi} \sin nu \sin mu \, du}_{\pi\delta_{mn}}$$

$$= \frac{l}{2} \sum_{n,m=1}^{\infty} B_n B_m^* e^{-i\frac{t}{\hbar}(E_n - E_m)} \delta_{mn}$$

$$= \frac{l}{2} \sum_{n=1}^{\infty} |B_n|^2 \stackrel{!}{=} 1$$

$\Rightarrow$

$$\frac{2}{l} = \sum_{n=1}^{\infty} |B_n|^2$$

Note that here Parseval's theorem could be exploited as well.