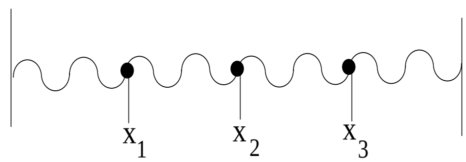
University College London
Department of Physics and Astronomy
2246E Mathematical Methods III
Coursework M2 (2007-2008)

Solutions to be handed in on Wednesday, November 21st, 2007

1. Three particles of equal masses, attached to a light spring, can move in a straight line, as illustrated in the diagram.



The equations pf motion may be written in matrix form

$$\frac{d^2\underline{x}}{dt^2} = \underline{A}\,\underline{x}\ ,$$

where $\underline{x}$ is a column vector of the displacements x_i and

$$\underline{A} = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix}\ .$$

Show that the eigenvalues of $\underline{A}$ are $\lambda_1 = -1$, $\lambda_2 = -3$ and $\lambda_3 = -4$. 5 MARKS

Find the corresponding normalized eigenvectors. 7 MARKS

2. A drumhead consists of a circular membrane attached to a rigid support along the circumference $r = a$. The vibrations are governed by the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial Z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2} = \frac{1}{v^2}\frac{\partial^2 Z}{\partial t^2}\ ,$$

where Z is the displacement from the equilibrium at polar coordinate (r, θ) and time t, and v is a constant. By assuming a solution of the form

Turn sheet over →

$$Z(r, \theta, t) = R(r) \times \Theta(\theta) \times T(t) \ ,$$

derive ordinary differential equations $R(r)$, $\Theta(\theta)$, and $T(t)$. 5 MARKS

Show that solutions with $Z = 0$ at $t = 0$ are of the form

$$Z = R_n(kr) \sin(kvt) \left[a_n \cos n\theta + b_n \sin n\theta \right] \ ,$$

where n is an integer. 8 MARKS

How can one find information on the possible values of k?

3. Prove that the second order differential equation

$$x^2 \frac{d^2y}{dx^2} + 2x^2 \frac{dy}{dx} - 2y = 0 \ ,$$

has a regular singular point at $x = 0$ and hence has two solutions of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+k} \ , \qquad a_0 \neq 0$$

with $k = -1$ or $k = 2$. 5 MARKS

By explicitly substituting the series expansion into the differential equation, show that for both series the ratio of neighbouring terms is given by

$$\frac{a_{n+1}}{a_n} = -\frac{2(n+k)}{(n+k+2)(n+k-1)} \ .$$

8 MARKS

Show that the series expansion for $k = -1$ solution terminates at $n = 1$ and verify explicitly that the resultant expression does satisfy the differential equation. 4 MARKS

Show that the $k = 2$ series converges for all values of x, 3 MARKS

and that the first two terms are proportional to the series expansion of

$$y = \left(1 + \frac{1}{x}\right) e^{-2x} + \left(1 - \frac{1}{x}\right) \ .$$

3 MARKS

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Model Answers

1. Solution to Problem 1

$$\underline{A} = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix}$$

$$\begin{aligned} |\underline{A} - \lambda \underline{I}| &= \begin{vmatrix} -3-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -3-\lambda \end{vmatrix} \\ &= -\begin{vmatrix} -(3+\lambda) & 0 \\ 1 & 1 \end{vmatrix} - (3+\lambda) \begin{vmatrix} -(3+\lambda) & 1 \\ 1 & -(2+\lambda) \end{vmatrix} \\ &= (3+\lambda)\left\{1 - [(3+\lambda)(2+\lambda) - 1]\right\} \\ &= (3+\lambda)\left[1 - \left(6 + 5\lambda + \lambda^2 - 1\right)\right] \\ &= (3+\lambda)\left[-4 - 5\lambda - \lambda^2\right] \\ &= -(3+\lambda)\left[\left(\lambda + \frac{5}{2}\right)^2 - \frac{9}{4}\right] = 0 \end{aligned}$$

hence

$$\lambda_1 = -3, \qquad \lambda_2 = -1, \qquad \lambda_3 = -4$$

For $\lambda_1 = -3$:

$$(\underline{A} - \lambda_1 \underline{I})\,\underline{x} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underline{0}$$

hence

$$\begin{aligned} x_2 &= 0 \\ x_3 &= -x_1 \end{aligned}$$

hence

$$\underline{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $\lambda_2 = -1$:

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -2x_1 + x_2 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned}$$

hence

$$\begin{aligned} x_2 &= 2x_1 \\ x_3 &= x_1 \end{aligned}$$

and therefore

$$\underline{v}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

For $\lambda_3 = -4$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underline{0}$$

therefore

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

hence

$$\begin{aligned} x_2 &= -x_1 \\ x_1 &= x_3 \end{aligned}$$

hence

$$\underline{v}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

2. Solution to Problem 2

$$\frac{1}{r}\partial_r\left(r\partial_r Z\right)+\frac{1}{r^2}\partial_\theta^2 Z=\frac{1}{v^2}\partial_t^2 Z$$

Set $Z=R(r)\Theta(\theta)T(t)$:

$$\frac{1}{r}\Theta T\partial_r\left(r\partial_r R\right)+\frac{1}{r^2}RT\frac{\partial_\theta^2\Theta}{\Theta}=\frac{1}{v^2}R\Theta\frac{\partial_t^2 T}{T}$$

divide equation bt $Z(theta,r,t)$:

$$\underbrace{\frac{1}{rR}\partial_r\left(r\partial_r R\right)+\frac{1}{r^2}\frac{\partial_\theta^2\Theta}{\Theta}}_{-k^2}=\underbrace{\frac{1}{v^2}\frac{\partial_t^2 T}{T}}_{-k^2}$$

therefore:

$$\partial_t^2=-k^2v^2T$$

and

$$T(t)=A\cos(kvt)+B\sin(kvt)$$

$Z=0$ at $t=0$, hence $A=0$

into left hand side:

$$\underbrace{\frac{r}{R}\partial_r(r\partial_r R)+k^2r^2}_{n^2}+\underbrace{\frac{\partial_\theta\Theta}{\Theta}}_{-n^2}=0$$

hence

$$\partial_\theta^2\Theta=-n^2\Theta$$

and

$$\Theta=A\cos n\theta+B\sin n\theta$$

but: $\Theta(theta)=\Theta(\theta+2\pi)$, therefore: n is integer!
Finally

$$\frac{r}{R}\partial_r(r\partial_r R)+k^2r^2=n^2$$

reparameterize $r\to kr$ or $\tilde{r}=kr$ then

$$\frac{\tilde{r}}{R}\partial_{\tilde{r}}(\tilde{r}\partial_{\tilde{r}}R)+\tilde{r}^2=n^2$$

and therefore $R_n(kr)$ solution! In summary:

$$R_n(kr)\sin(kvt)\left(a_n\cos n\theta+b_n\sin n\theta\right)$$

Boundary condition: $R_n(ka)=0$.

3. Solution to Problem 3

$$x^2y'' + 2x^2y' - 2y = 0$$

or

$$y'' + 2y' - \frac{2}{x^2}y = 0$$

and

$$\begin{aligned} p(x) &= 2 \\ q(x) &= = -\frac{2}{x^2} \end{aligned}$$

therefore

$$p_0 = \lim_{x\to 0} xp(x) = 0$$
$$q_0 \lim_{x\to 0} x^2 q(x) = -2$$

hence em regular singular point at $x = 0$.
Indicial equation $k(k-1) + p_0 k + q_0 = 0$:

$$\begin{aligned} k(k-1) - 2 &= 0 \\ k^2 - k - 2 &= 0 \\ \left(k - \frac{1}{2}\right)^2 &= \frac{9}{4} \end{aligned}$$

and therefore $k_1 = 2$ and $k_2 = -1$.

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^{n+k} \\ y'(x) &= \sum_{n=0}^{\infty} a_n (n+k) x^{n+k-1} \\ y''(x) &= \sum_{n=0}^{\infty} a_n (n+k)(n+k-1) x^{n+k-2} \end{aligned}$$

Hence

$$\sum_{n=0}^{\infty} a_n(n+k)(n+k-1)x^{n+k} + 2\sum_{n=0}^{\infty} a_n(n+k)x^{n+k+1} - 2\sum_{n=0}^{\infty} a_n x^{n+k} = 0$$

In the second term replace $n+1 \to n$ and obtain:

$$\sum_{n=0}^{\infty} x^{n+k} a_n \left[(n+k)(n+k-1)-2\right] + 2\sum_{n=1}^{\infty} a_{n-1}(n+k-1)x^{n+k}$$

for $n=0$:

$$k(k-1)-2=0 \checkmark$$

is fullfilled from indicial equation.
Remaining terms

$$\sum_{n=1}^{\infty} x^{n+k} \left\{a_n \left[(n+k)(n+k-1)-2\right] + 2a_{n-1}(n+k-1)\right\} = 0$$

and we obtain

$$a_n \left[(n+1+k)(n+k)-2\right] = -2a_{n-1}(n+k-1)$$

Replace $n \to n+1$:

$$a_{n+1} = -2\frac{2(n+k)}{(n+k+2)(n+k-1)}a_n$$

For $k=-1$ we obtain

$$a_{n+1} = -\frac{2(n-1)}{(n+1)(n-2)}a_n$$

For $n=0$: $a_1 = -a_0$
For $n=1$: $a_2 = 0$ and all other $a_n = 0$.
Hence

$$y = x^{-1}\left[a_0 - a_0 x\right] = a_0\left[x^{-1} - 1\right]$$

Hence

$$\begin{aligned} y' &= a_0\frac{-1}{x^2} \\ y'' &= 2a_0\frac{1}{x^3} \end{aligned}$$

in DEQN:

$$\frac{2a_0}{x} - 2a_0 - 2\frac{a_0}{x} + 2a_0 = 0 \checkmark$$

For $k=2$:

$$\frac{a_{n+1}}{a_n} = -\frac{2(n+2)}{(n+4)(n+1)}$$

Radius of convergence:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{x_n x^n}\right| = x^2 \lim_{n\to\infty}\frac{n+2}{(n+4)(n+1)} \to 0 < 1$$

so convergence for all x.
For two terms: $a_1 = -a_0$ and $a_2 = 6a_0/10$. Hence

$$y \approx x^2 a_0 \left[1 - x + \frac{6}{10}x^2\right]$$

From

$$y = \left(1 + \frac{1}{x}\right) e^{-2x} + 1 - \frac{1}{x} \approx \left(1 + \frac{1}{x}\right)\left[1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{16}{3}x^4\right] + 1 - \frac{1}{x} = \frac{2}{3}x^2 - \frac{2}{3}x^3$$

which is the same as $a_0 x^2(1-x)$ for $a_0 = 2/3$.