Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-304 PHYSICAL DYNAMICS

Time Allowed: 2 hours 30 minutes

Date: 24th May, 2010

Time: 10:00 - 12:30

Instructions: **Answer ALL questions in section A. Answer ONLY TWO questions from section B. Section A carries 50 marks, each question in section B carries 25 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. Course work comprises 25% of the final mark.**

A formula sheet containing mathematical results that may be of help in various questions is provided at the end of the examination paper.

Numeric calculators are permitted in this examination. Please state on your answer book the name and type of machine used. Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important Note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from the college. Please check now to ensure that you do not have any notes in your possession. If you have any then please raise your hand and give them to an invigilator immediately. Exam papers cannot be removed from the exam room

You are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Examiners: Dr G Travaglini, Dr R Russo

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SECTION A. Attempt answers to all questions.

A1. Define what a conservative force is. **[3]**

A2. Consider a single particle of mass m moving in one dimension parameterised by the coordinate x. The particle is subject to a conservative force whose potential is $V(x)$. Write down the expression for the energy and prove, either by an explicit calculation or by using Noether's theorem, that it is a conserved quantity. **[5]**

A3. Define what is meant by an equilibrium position of the system described in A2. **[3]**

A4. Write down the Lagrangian and the Lagrange equations of the system described in part A2. **[5]**

A5. Define what is meant by the number of degrees of freedom of a mechanical system. **[3]**

A6. Explain what is meant by a set of generalised coordinates for a mechanical system. **[3]**

A7. Consider the Lagrangian $L(q_1, q_2; \dot{q}_1, \dot{q}_2)$ of a conservative mechanical system described by generalised coordinates (q_1, q_2). What condition must the Lagrangian satisfy in order for the momentum p_1 associated to q_1 to be conserved? **[4]**

A8. For the system described in A7, what would the Noether symmetry responsible for the conservation of p_1 be? **[3]**

A9. A conservative mechanical system with one degree of freedom is described by a single generalised coordinate q. Explain how the Hamiltonian H of the system is obtained from the Lagrangian, and write down its expression. **[5]**

A10. Consider a system of N pointlike particles of masses m_i, $i = 1, \ldots, N$ with coordinates $\mathbf{r}_i$ with respect to an inertial frame. Define the position $\mathbf{R}$ of the centre of mass of this system. **[3]**

A11. Explain what is meant by the centre of mass frame for the system described in A10. **[4]**

A12. Explain under which conditions the system considered in part A10 is a rigid body. **[4]**

A13. Explain briefly how many degrees of freedom a rigid body that moves in a three-dimensional space has, and what motions these degrees of freedom describe. **[5]**

SECTION B. Answer two of the four questions in this section.

B1

A conservative mechanical system consists of a spring of rest length l_0, negligible mass and spring constant k, and a pointlike mass m attached to one end of the spring. The other end of the spring is fixed at point O (see figure, where the spring is represented by a zig-zag line). The system is free to oscillate in a vertical plane about the point O. Gravity acts along the vertical direction, and the force exerted by the spring on the mass is, as usual, $F = -k(r - l_0)$, where r is the distance between the position of the mass m and the point O.

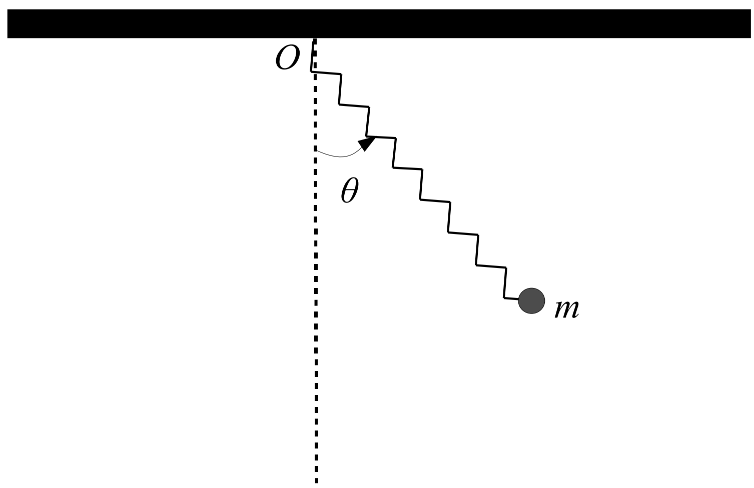
(i) How many degrees of freedom does the system have? **[4]**

(ii) Choosing appropriate generalised coordinates, write down the Lagrangian of the system and the Lagrange equations. **[8]**

(iii) Calculate the values r_0 and θ_0 of r and θ, respectively, at the equilibrium positions (the angle θ is indicated in the figure). **[4]**

(iv) Discuss the stability of the equilibrium positions obtained in part (iii). **[4]**

(v) Find the frequencies of small oscillations about the stable equilibrium position determined above. **[5]**



B2

A conservative mechanical system consists of two particles of masses m_1 and m_2 interacting through a central force whose potential is $V(r)$. Here, $r := |\mathbf{r}_1 - \mathbf{r}_2|$, and $\mathbf{r}_1$ and $\mathbf{r}_2$ denote the positions of the two particles in an inertial reference frame $(\hat{x}, \hat{y}, \hat{z})$ and origin at O.

(i) Write down the Lagrangian of the system. **[5]**

(ii) Introducing the centre of mass coordinate $\mathbf{R}$ and the relative coordinate $\mathbf{r} := \mathbf{r}_1 - \mathbf{r}_2$, show that the Lagrangian of the system in terms of the generalised coordinates $\mathbf{R}$, $\mathbf{r}$ and their time derivatives $\dot{\mathbf{R}}$ and $\dot{\mathbf{r}}$ is equal to $L = L_{\text{free}}(\dot{\mathbf{R}}) + L_{\text{rel}}(\mathbf{r}, \dot{\mathbf{r}})$, where $L_{\text{free}} = (1/2)(m_1 + m_2)\dot{\mathbf{R}}^2$, and $L(\mathbf{r}, \dot{\mathbf{r}}) = (1/2)\mu\dot{\mathbf{r}}^2 - V(r)$, where $\mu := m_1 m_2/(m_2 + m_2)$ is the reduced mass. Explain why the motion of the centre of mass is trivial. **[6]**

(iii) From now on we focus only on the Lagrangian describing the relative motion, i.e. we study the motion of a fictitious particle of mass μ whose Lagrangian is $L_{\text{rel}}(\mathbf{r}, \dot{\mathbf{r}})$ introduced in part (ii). Show that the relative angular momentum $\mathbf{L}' := \mathbf{r} \times \mathbf{p}$ is conserved. Here $\mathbf{p} := \mu\dot{\mathbf{r}}$. **[4]**

(iv) The direction of the conserved vector $\mathbf{L}'$ determines the plane of the orbit. Choosing plane polar coordinates (r, ϕ) on this plane as generalised coordinates, rewrite the Lagrangian L_{rel} in terms of these coordinates, and use the Lagrange equations to show that the momentum p_ϕ associated to ϕ is a constant of motion. What is its physical interpretation? **[5]**

(v) Write down the expression for the energy of the fictitious particle of mass μ, and use the conservation of p_ϕ to eliminate ϕ from this expression, thus giving the energy in the form $E(r, \dot{r}) = (1/2)\mu\dot{r}^2 + V_{\text{eff}}(r)$. Calculate the one-dimensional effective potential $V_{\text{eff}}(r)$. **[5]**

B3

A mechanical system consists of a rigid rod of length l and mass M, and a pointlike mass m attached to the rod at a distance d from the extremum O of the rod, see the figure below. The mass per unit length of the rod is constant. The system can move in a vertical plane $(\hat{x}, \hat{y})$ with the extremum O fixed, so that the system is free to oscillate in the vertical plane about the point O. Gravity acts along the vertical direction.

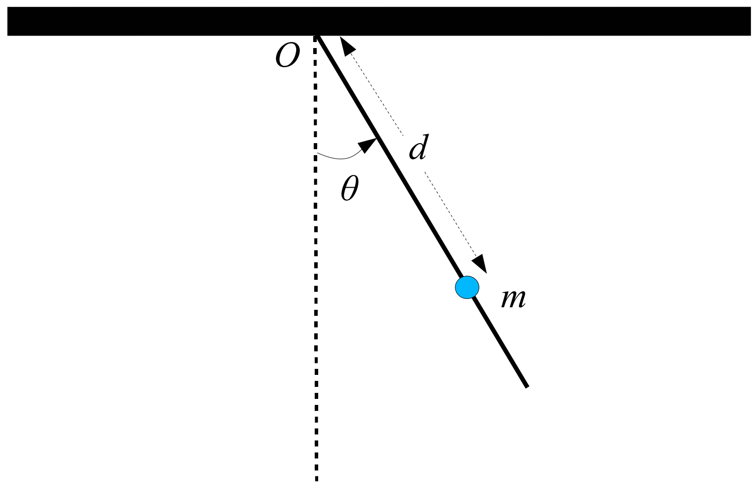
(i) How many degrees of freedom does the system have? **[4]**

(ii) Calculate the moment of inertia of the system constituted by the rod and the pointlike mass with respect to an axis orthogonal to the plane $(\hat{x}, \hat{y})$ and passing through O. Assume that the width of the rod is negligible compared to its length, so that you can effectively treat it as one-dimensional. **[5]**

(iii) Determine the distance of the centre of mass of the system from the point O, and write down the gravitational potential of the system. **[5]**

(iv) Write down the Lagrangian of the system and the Lagrange equations. **[6]**

(v) Determine the frequency of small oscillations about the equilibrium position $\theta = 0$ (the angle θ is defined in the figure). **[5]**



B4

Consider the Lagrangian of the one-dimensional harmonic oscillator,

$$L \;=\; \frac{m}{2}\dot{x}^2 \;-\; \frac{m}{2}\omega^2 x^2 \ .$$

(i) Write down the Lagrange equation for the system. **[4]**

(ii) Write down the Hamiltonian of the system, and the corresponding Hamilton equations. **[5]**

(iii) Consider the complex quantities

$$a \;:=\; \sqrt{\frac{m\omega}{2}}\Big(x + \frac{ip}{m\omega}\Big) \ , \qquad a^* \;:=\; \sqrt{\frac{m\omega}{2}}\Big(x - \frac{ip}{m\omega}\Big) \ .$$

Calculate aa^* and re-express the Hamiltonian in terms of a and a^*. **[4]**

(iv) Calculate the Poisson brackets $\{a, a^*\}$, $\{a, H\}$, $\{a^*, H\}$, where for any two given functions of x and p, $A(x,p)$, $B(x,p)$, the Poisson bracket $\{A, B\}$ is defined as

$$\{A, B\} := \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x} \ .$$

[6]

(v) Write down the time evolution equation for $a(t)$, and its solution. **[6]**

FORMULA SHEET

Plane polar coordinates:

$$
\begin{aligned}
\mathbf{r} &:= r\,\hat{\mathbf{r}}\ , \\
\dot{\mathbf{r}} &= \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\phi}\,\hat{\boldsymbol{\phi}}\ , \\
\dot{\mathbf{r}}^2 &= \dot{r}^2 + r^2\,\dot{\phi}^2\ .
\end{aligned}
$$

