SECTION A

A1. $\vec{F}$ is conservative if it can be written as the gradient of a function of the coordinates

$$\boxed{\vec{F} = -\vec{\nabla} V(\vec{r})}$$

potential

A2. $$\boxed{E(x,\dot{x}) = \frac{1}{2} m \dot{x}^2 + V(x)}$$

The Lagrangian does not depend explicitly on time. By Noether's theorem, the energy is conserved.
Or, with a direct calculation:

$$\dot{E} = \frac{\partial E}{\partial x}\dot{x} + \frac{\partial E}{\partial \dot{x}}\ddot{x} = V'(x)\,\dot{x} + m\dot{x}\ddot{x} =$$

$$= \dot{x}\underbrace{\left(V'(x) + m\ddot{x}\right)}_{=0 \text{ by Newton's equations}} = 0$$

A3. An equilibrium position x_0 for the one-dimensional system in point A2 is a position where the force $F(x) = -V'(x)$ vanishes. I.e. x_0 is a solution to $\boxed{V'(x_0) = 0}$ (i.e. an extremum for V)

A4. $$\boxed{L = \frac{1}{2} m \dot{x}^2 - V(x)}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \Rightarrow \boxed{m\ddot{x} = -V'(x)}$$

A5. The <u>number of degrees of freedom</u> of a mechanical system is the number of independent parameters that have to be specified in order to determine completely the position of the system.
[In mathematical terms: it is the dimension of the configuration space of the system.]

A6. A set of <u>generalised coordinates</u> is a set of parameters $\vec{q} = (q_1, \dots, q_N)$ (where N = number of degrees of freedom) which determine completely the position of the system.

A7. From Lagrange's equations for q_1

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial q_1} \quad \text{or} \quad \dot{p}_1 = \frac{\partial L}{\partial q_1}$$

$\Rightarrow$ p_1 is conserved if $\frac{\partial L}{\partial q_1} = 0$ i.e. if

<u>L is independent of q_1.</u>

A8. Translations in the q_1 direction.

A9. $\boxed{H = p\dot{q} - L}$ where $p \equiv \frac{\partial L}{\partial \dot{q}}(q, \dot{q})$

and $\dot{q} = \dot{q}(q, p)$ through the equation which defines p.

A.10 The position of the centre of mass $\vec{R}$ is defined as

$$\boxed{\vec{R} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i}}$$

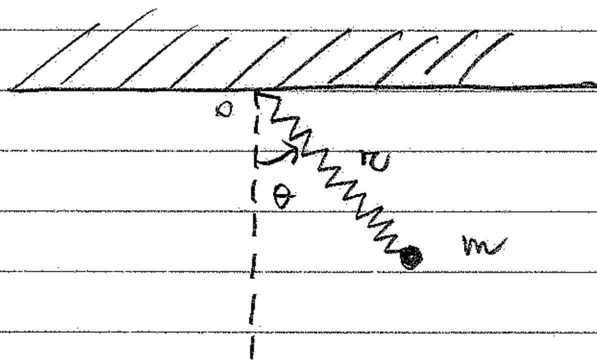
A.11 A centre of mass frame is a coordinate system whose origin is set to be in the centre of mass O' of the system.

A.12 In order for the system of N particles to be a rigid body, all the relative distances between particles must be time-independent, that is the quantities $|\vec{r}_i(t) - \vec{r}_j(t)| \equiv C_{ij}$ do not depend on t, $\forall\, i$ and j.

A.13 Six degrees of freedom. Three are translational; they fix the position of one point of the body (for example its centre of mass). The remaining three are rotational; they describe the orientation of the body in three-dimensional space (Euler's angles are a particularly convenient common choice)

SECTION B

PROBLEM B1



(i) The system has 2 degrees of freedom. We can choose plane polar coordinates (r, θ) in the (vertical) plane containing the system as generalised coordinates.

(ii) $L = T - V$

The kinetic energy of the mass is

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2 \dot{\theta}^2\right)$$

The potential is the sum of two terms,

$V = V_{spring} + V_{gravity}$ where

$$V_{spring} = \frac{1}{2} k (r - l_0)^2 \quad , \quad V_{gravity} = - m g r \cos\theta$$

hence

$$\boxed{L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{2} k (r - l_0)^2 + m g r \cos\theta}$$

The Lagrange equations:

1) $\frac{d}{dt}\frac{\partial L}{\partial \dot r} = \frac{\partial L}{\partial r}$ gives

$$\boxed{m\ddot r = -k(r-l_0) + mg\cos\theta + mr\dot\theta^2}$$

2) $\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = \frac{\partial L}{\partial \theta}$ gives

$$mr^2\ddot\theta + 2mr\dot r\dot\theta = -mgr\sin\theta \quad \text{or}$$

$$\boxed{r\ddot\theta = -g\sin\theta - 2\dot r\dot\theta}$$

(iii) Equilibrium position:

at $\theta = 0$ we get $-k(r_0 - l_0) + mg = 0$

hence $\boxed{r_0 = l_0 + m\frac{g}{k}}$

(iv) Stability: $V(r,\theta) = \frac{1}{2}k(r-l_0)^2 - mgr\cos\theta$

$$\begin{cases} \frac{\partial V}{\partial r} = k(r-l_0) - mg\cos\theta \\ \frac{\partial V}{\partial \theta} = mgr\sin\theta \end{cases}$$

$$\frac{\partial^2 V}{\partial r^2} = k, \quad \frac{\partial^2 V}{\partial \theta^2} = mgr\cos\theta, \quad \frac{\partial^2 V}{\partial r\partial\theta} = mg\sin\theta$$

At the equilibrium position $(r_0, \theta=0)$ the matrix of the 2^{nd} derivatives is

$$V^{(2)} = \begin{pmatrix} k & 0 \\ 0 & mgr_0 \end{pmatrix}$$

$\Rightarrow$ $(r_0, 0)$ is a stable equilibrium position (as it was clear physically)

(v) We need to solve the secular equation $\boxed{\det\left(V^{(2)} - \omega^2 T^{(2)}\right) = 0}$

where $V^{(2)}$ and $T^{(2)}$ are the matrices of the 2^{nd} derivatives of the potential and kinetic energy evaluated at the equilibrium, with $T^{(2)} = \begin{pmatrix} m & 0 \\ 0 & mr_0^2 \end{pmatrix}$ and $V^{(2)}$ given above.

$\Rightarrow$ $\det\begin{pmatrix} k-\omega^2 m & 0 \\ 0 & mgr_0 - \omega^2 m r_0^2 \end{pmatrix} = 0$ $\Rightarrow$

$$\boxed{\omega_1^2 = k/m, \qquad \omega_2^2 = g/r_0}$$

with $r_0 = l_0 + m\dfrac{g}{k}$ determined earlier.

PROBLEM B2

(i) $$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - V(r)$$

where $r = |\vec{r}_1 - \vec{r}_2|$

(ii) Introducing $\vec{R} = \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$, $\vec{r} = \vec{r}_1 - \vec{r}_2$

we get

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1+m_2} \vec{r}, \quad \vec{r}_2 = \vec{R} - \frac{m_1}{m_1+m_2} \vec{r}$$

$$\dot{\vec{r}}_1^{\,2} = \dot{\vec{R}}^2 + \frac{m_2^2}{(m_1+m_2)^2} \dot{\vec{r}}^{\,2} + \frac{2 m_2}{m_1+m_2} \dot{\vec{R}} \cdot \dot{\vec{r}}$$

$$\dot{\vec{r}}_2^{\,2} = \dot{\vec{R}}^2 + \frac{m_1^2}{(m_1+m_2)^2} \dot{\vec{r}}^{\,2} - \frac{2 m_1}{m_1+m_2} \dot{\vec{R}} \cdot \dot{\vec{r}}$$

$$\Rightarrow \frac{1}{2}\left(m_1 \dot{\vec{r}}_1^{\,2} + m_2 \dot{\vec{r}}_2^{\,2}\right) = \frac{1}{2}(m_1+m_2) \dot{\vec{R}}^2 + \frac{1}{2} \frac{m_1 m_2}{m_1+m_2} \dot{\vec{r}}^{\,2}$$

Hence

$$L = \underbrace{\frac{1}{2}(m_1+m_2) \dot{\vec{R}}^2}_{L_{free}(\dot{\vec{R}})} + \underbrace{\frac{1}{2} \mu \dot{\vec{r}}^{\,2} - V(r)}_{L_{rel}(\vec{r}, \dot{\vec{r}})}$$

The motion of the centre of mass $\vec{R}$ is solely determined by a trivial, free Lagrangian. Its motion is decoupled from the relative motion described by $L_{rel}(\vec{r}, \dot{\vec{r}})$.

(iii) In order to show that $\vec{L}'$ is conserved we can use Noether's theorem.

$L_{rel} = \frac{1}{2}\mu\dot{\vec{r}}^2 - V(|\vec{r}|)$ is invariant under three-dimensional rotations of $\vec{r}$ $\Rightarrow$ $\vec{L}'$ is conserved.

Alternatively, by a direct calculation, we get

$$\dot{\vec{L}}' = \vec{r} \times \underbrace{\dot{\vec{p}}}_{\vec{F}} + \underbrace{\dot{\vec{r}} \times \vec{p}}_{=0} = \vec{r} \times \vec{F} = \vec{0}$$

since the force is central, i.e. $\vec{F} \parallel \vec{r}$.

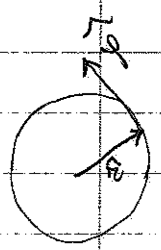
(iv) Using plane polar coordinates (r, φ), we have

$$\boxed{L_{rel} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r)}$$

Lagrange's equation for φ gives

$$\boxed{\dot{p}_\varphi = 0} \qquad \text{where } p_\varphi \equiv \frac{\partial L}{\partial \dot{\varphi}} = \mu r^2 \dot{\varphi}$$

$|p_\varphi|$ is the modulus of $\vec{L}'$:



$$\vec{r} \times \dot{\vec{r}} = \vec{r} \times r \underbrace{\frac{d}{dt}(\hat{r})}_{\equiv \dot{\varphi}\hat{\varphi}} = r^2\dot{\varphi}\,\underbrace{(\hat{r} \times \hat{\varphi})}_{\hat{z}} \Rightarrow$$

$$\Rightarrow |\vec{L}'| = \mu r^2 |\dot{\varphi}|.$$

(V) $E = T + V =$

$$= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + V(r)$$

Using $\dot{\varphi} = p_\varphi/(\mu r^2)$ we get

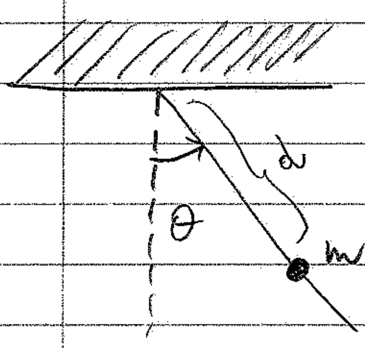
$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{p_\varphi^2}{2\mu r^2} + V(r) =$$

$$= \frac{1}{2}\mu\dot{r}^2 + V_{eff}(r)$$ with

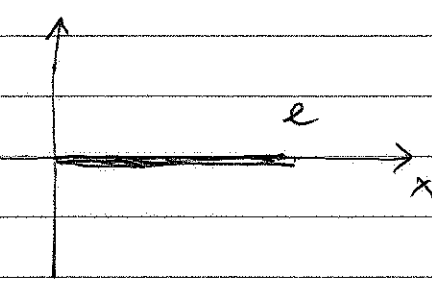
$$\boxed{V_{eff}(r) = V(r) + \frac{p_\varphi^2}{2\mu r^2}}$$

PROBLEM B3

(i) The system has 1 degree of freedom parametrised by the angle θ in the figure



(ii) The moment of inertia of the rod alone is



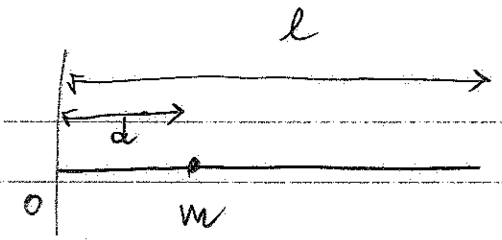
$$I_{rod} = \int_0^{\ell} dx\, \rho x^2$$

Since the density is constant, $\rho = M/\ell$

and $I_{rod} = \frac{M}{\ell}\int_0^{\ell} dx\, x^2 = \frac{1}{3}M\ell^2$

Additivity of I implies that the total moment of inertia is now

$$\boxed{I = \frac{1}{3}M\ell^2 + md^2}$$

(iii) 

The distance from O of the centre of mass is given by

$$\frac{1}{M+m}\left(\frac{M}{\ell}\left(\frac{\ell^2}{2}\right)+md\right)=\frac{md+M\frac{\ell}{2}}{M+m}$$

where we have used the definition of centre of mass

$$\vec{R}_{c.o.m.}=\frac{\sum m_i\vec{r}_i}{\sum m_i}$$

The gravitational potential will therefore be

$$V(\theta)=-(M+m)\frac{\left(md+M\frac{\ell}{2}\right)}{M+m}g\cos\theta \Rightarrow$$

$$\boxed{V(\theta)=-\left(md+M\frac{\ell}{2}\right)g\cos\theta}$$

as one could have perhaps intuitively guessed.

(IV) $\boxed{L=\frac{1}{2}I\dot{\theta}^2-V(\theta)}$ with I and V

written above. The Lagrange equations:

$I\ddot{\theta}=-V'(\theta)$. In our case

$$\left(\frac{1}{3}Ml^2 + md^2\right)\ddot{\theta} = -\left(md + M\frac{l}{2}\right)g\sin\theta$$

or

$$\boxed{\ddot{\theta} + \frac{md + M\frac{l}{2}}{md^2 + \frac{1}{3}Ml^2}\, g\sin\theta = 0}$$

(v) In the small oscillations approximation we expand $\sin\theta$ about the equilibrium position $\theta = 0$, $\sin\theta \simeq \theta + O(\theta^3)$

The equation is of the form

$\ddot{\theta} + \omega_{s.o.}^2\,\theta = 0$ $\Rightarrow$ the frequency of small oscillations is

$$\boxed{\omega_{s.o.}^2 = \frac{md + M\frac{l}{2}}{md^2 + \frac{1}{3}Ml^2}\, g}$$

PROBLEM B4

(i) $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \Rightarrow m\ddot{x} = -m\omega^2 x$

or $\boxed{\ddot{x} + \omega^2 x = 0}$

(ii) $H = p\dot{x} - L$ with

$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow$

from $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$ we get

$H = \frac{p^2}{m} - \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}m\omega^2 x^2$ or

$$\boxed{H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2}$$

The Hamilton equations are

$\dot{x} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial x}$ or

$$\boxed{\dot{x} = \frac{p}{m}, \quad \dot{p} = -m\omega^2 x}$$

(iii) We notice that

$$aa^* = \frac{m\omega}{2}\left(X^2 + \frac{p^2}{m^2\omega^2}\right) =$$

$$= \frac{1}{\omega}\left[\frac{p^2}{2m} + \frac{1}{2}m\omega^2X^2\right] \Rightarrow$$

$$\boxed{aa^* = \frac{1}{\omega}\left[\frac{p^2}{2m} + \frac{1}{2}m\omega^2X^2\right]}$$ hence

$$\boxed{H = \omega aa^*}$$

(iv) The Poisson bracket of $\{A, B\}$ is

$$\{A, B\} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x}$$

Here

$$\frac{\partial a}{\partial x} = \sqrt{\frac{m\omega}{2}} = \frac{\partial a^*}{\partial x}$$

$$\frac{\partial a}{\partial p} = \frac{i}{\sqrt{2m\omega}} = -\frac{\partial a^*}{\partial p} \Rightarrow$$

$$\{a, a^*\} = -\frac{i}{2} - \frac{i}{2} = -i$$

$$\boxed{\{a, a^*\} = -i}$$

Then $\{a, H\} = \omega \{a, a^*\} a = -i\omega a$

Hence $$\boxed{\begin{aligned} \{a, H\} &= -i\omega a \\ \{a^*, H\} &= +i\omega a^* \end{aligned}}$$

(v) The time evolution of a function $A(X, p)$ is

$$\dot{A} = \frac{\partial A}{\partial x}\dot{x} + \frac{\partial A}{\partial p}\dot{p} = \frac{\partial A}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial x} =$$

$$= \{A, H\}$$

Hence the time evolution for $a(t)$ is

$$\dot{a}(t) = \{a, H\} \Rightarrow \boxed{\dot{a}(t) = -i\omega a(t)}$$

whose solution is $\boxed{a(t) = a(0)\, e^{-i\omega t}}$