UNIVERSITY COLLEGE LONDON
DEPARTMENT OF PHYSICS AND ASTRONOMY

2246 MATHEMATICAL METHODS III

Problem Sheet M7 (2011–12)

Problems sheets will not be marked. The mark allocation shown will however be relevant to in-course tests (which will be based on a selection of questions from weekly Problem Sheets and Tutorial Problem Sheets).

1. The Legendre polynomials can also be obtained by the following relation ('Rodrigues formula'):

$$P_l(x) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l \, .$$

Verify this formula for the first three polynomials (for $l = 0, 1, 2$), comparing the expressions obtained to those given in the lectures. **[3 mark]**

Using Rodrigues formula, show that **[5 mark]**

$$\int_{-1}^{1} x^m P_l(x)\,\mathrm{d}x = 0 \quad \text{if} \quad m < l \, .$$

Now, infer the orthogonality relation for the Legendre polynomials from the previous equation. **[2 mark]**

2. The gravitational potential of a mass M located on the z axis at distance d from the origin is given, in spherical polar coordinates, by (G is the gravitational constant):

$$V(r,\theta,\phi) = -\frac{GM}{r}\sum_{l=0}^{\infty} P_l(\cos\theta)\frac{d^l}{r^l} \, .$$

Consider now a second mass m very far from the origin ($d \ll r$). Approximating the potential due to mass M to the first three terms of the previous expansion, and using the expression for the gradient in spherical coordinates, find the gravitational force acting on the mass m. **[10 mark]**

Reminder: the gradient operator in spherical polar coordinates is given by (see lecture notes, Chapter 1):

$$\nabla = \hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \, .$$

1. Check: **[3]**

$$
\begin{aligned}
P_0 &= \frac{1}{2^0 0!}(x^2-1)^0 = 1 \, , \\
P_1 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1) = x \, , \\
P_2 &= \frac{1}{8}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^4-2x^2+1) = \frac{1}{8}\frac{\mathrm{d}}{\mathrm{d}x}(4x^3-4x) = \frac{1}{2}(3x^2-1) \, . \qquad (1)
\end{aligned}
$$

Proof of orthogonality via Rodrigues formula:
First, let us notice that any derivative of order $k < l$ of $(x^2-1)^l$ is proportional to (x^2-1) (one can easily convince oneself by differentiating it once, and then verifying that, in further differentiations, a factor (x^2-1) is always present in all terms). This implies that such derivatives always vanish for $x = \mp 1$, in formulae:

$$
\frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^2-1)_{|x=\mp 1} = 0 \, , \quad \text{for } k < l. \qquad (2)
$$

Now, inserting Rodrigues formula into the integral and integrating by parts m times yields (for $m < l$)

$$
\begin{aligned}
\int_{-1}^{+1} x^m P_l \mathrm{d}x &= \frac{1}{2^l l!}\int_{-1}^{+1} x^m \frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l \mathrm{d}x \\
&= \frac{1}{2^l l!}\left(\left[x^m \frac{\mathrm{d}^{l-1}}{\mathrm{d}x^{l-1}}(x^2-1)^l\right]_{-1}^{+1} - m\int_{-1}^{+1} x^{m-1}\frac{\mathrm{d}^{l-1}}{\mathrm{d}x^{l-1}}(x^2-1)^l \mathrm{d}x\right) \\
&\vdots \\
&= \frac{1}{2^l l!}\left(\sum_{k=0}^{m-1}(-1)^k\left[x^{m-k}\frac{\mathrm{d}^{l-k-1}}{\mathrm{d}x^{l-k-1}}(x^2-1)^l\right]_{-1}^{+1} + (-1)^m m! \int_{-1}^{+1}\frac{\mathrm{d}^{l-m}}{\mathrm{d}x^{l-m}}(x^2-1)^l \mathrm{d}x\right)
\end{aligned}
$$

The terms in the sum come from the integrations by part: because of Eq. (2) all such terms vanish, so that the integral reduces to **[5]**

$$
\int_{-1}^{+1} x^m P_l \mathrm{d}x = \frac{(-1)^m m!}{2^l l!}\int_{-1}^{+1}\frac{\mathrm{d}^{l-m}}{\mathrm{d}x^{l-m}}(x^2-1)^l \mathrm{d}x = \left[\frac{\mathrm{d}^{l-m-1}}{\mathrm{d}x^{l-m-1}}(x^2-1)^l\right]_{-1}^{+1} = 0
$$

(again, due to Eq. (2)). Notice that the last expression is well defined because we assumed $m < l$.

Let us then consider two distinct Legendre polynomials P_l and P_m assuming, without loss of generality, $m < l$. P_m can be written as $\sum_{k=0}^{m} p_k x^k$ for some real coefficients p_k. Therefore: **[2]**

$$\int_{-1}^{+1} P_l P_m \mathrm{d}x = \sum_{k=0}^{m} p_k \int_{-1}^{+1} x^k P_l \mathrm{d}x = 0$$

because of the previously shown relation (remember that $k \leq m < l$ by assumption).

2. The potential energy $E(r, \theta, \phi)$ of the mass m in the gravitational field of the mass M is given by **[2]**

$$E(r, \theta, \phi) = mV(r, \theta, \phi) = -\frac{GMm}{r} \sum_{l=0}^{\infty} P_l(\cos\theta) \frac{d^l}{r^l} \ ,$$

which, for large r, can be approximated as (see expression for first three Legendre polynomials) **[3]**

$$E(r, \theta, \phi) \simeq -\frac{GMm}{r} \left(1 + \cos\theta \frac{d}{r} + \frac{1}{2}(3(\cos\theta)^2 - 1)\frac{d^2}{r^2} \right) \ .$$

Gradient in spherical polars

$$\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \ .$$

The gravitational force $\underline{F}$ acting on the mass m is then **[5]**

$$\begin{aligned}
\underline{F} &= -\nabla E(r, \theta, \phi) \simeq \\
& GMm \left[\hat{\mathbf{e}}_r \frac{\partial}{\partial r} \left(\frac{1}{r} + \cos\theta \frac{d}{r^2} + \frac{1}{2}(3(\cos\theta)^2 - 1)\frac{d^2}{r^3} \right) \right. \\
& \left. + \hat{\mathbf{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \left(\frac{1}{r} + \cos\theta \frac{d}{r^2} + \frac{1}{2}(3(\cos\theta)^2 - 1)\frac{d^2}{r^3} \right) \right] \\
&= -GMm \left[\hat{\mathbf{e}}_r \left(\frac{1}{r^2} + 2\cos\theta \frac{d}{r^3} + \frac{3}{2}(3(\cos\theta)^2 - 1)\frac{d^2}{r^4} \right) + \hat{\mathbf{e}}_\theta \left(\sin\theta \frac{d}{r^3} + 3\sin\theta\cos\theta \frac{d^2}{r^4} \right) \right] .
\end{aligned}$$