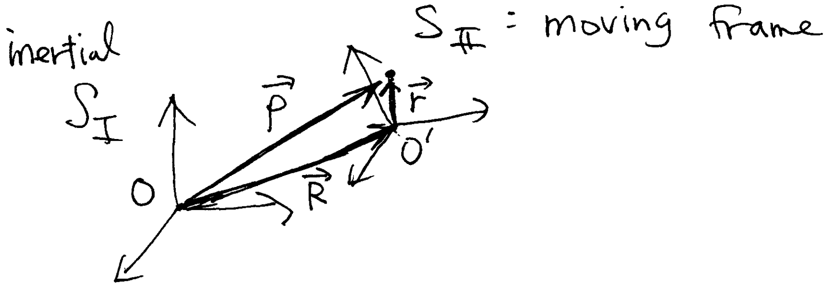
6. Motion in a non-inertial frame



$$\underbrace{\vec{\rho}}_{\text{coordinates in inertial frame}} = \vec{R} + \underbrace{\vec{r}}_{= r_a \underbrace{\vec{f}^{(a)}}_{\text{moving basis}} \text{ in the moving frame}}$$

Newton's law says that, for the inertial $\vec{\rho}$,

$$m\ddot{\vec{\rho}} = \vec{F}$$

but what does this look like to a <u>moving</u> observer? Namely, we want to know the time evolution of the coordinates in the moving frame, r_a.

Let us work it out.

$$\dot{\vec{\rho}} = \dot{\vec{R}} + \dot{\vec{r}}$$

Her

$$\dot{\vec{r}} = \dot{r}_a \vec{f}^{(a)} + r_a \dot{\vec{f}}^{(a)}$$

Now, use

$$\dot{\vec{f}}^{(a)} = \underbrace{\vec{\omega}}_{\substack{\text{instantaneous} \\ \text{angular velocity of the moving frame} \\ \text{about its origin } O'}} \times \vec{f}^{(a)}$$

$$\Rightarrow \dot{\vec{r}} = \dot{r}_a \vec{f}^{(a)} + \underbrace{r_a \vec{\omega} \times \vec{f}^{(a)}}_{\vec{\omega} \times \vec{r}}$$

One more dot:

$$\ddot{\vec{\rho}} = \ddot{\vec{R}} + \ddot{\vec{r}}$$

$$\ddot{\vec{r}} = \frac{d}{dt}\left(\dot{r}_a \vec{f}^{(a)} + \vec{\omega} \times \vec{r}\right)$$

$$= \ddot{r}_a \vec{f}^{(a)} + \dot{r}_a \underbrace{\dot{\vec{f}}^{(a)}}_{\vec{\omega} \times \vec{f}^{(a)}} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \underbrace{\dot{\vec{r}}}_{\dot{r}_a \vec{f}^{(a)} + \vec{\omega} \times \vec{r}}$$

$$= \ddot{r}_a \vec{f}^{(a)} + 2\vec{\omega} \times \dot{r}_a \vec{f}^{(a)} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Now define

$$\dot{r}_a \vec{f}^{(a)} \equiv \vec{v} \quad : \text{ velocity measured in } S_{II}$$

$$\ddot{r}_a \vec{f}^{(a)} \equiv \vec{a} \quad : \text{ acceleration } \quad ''$$

Then

$$\boxed{\ddot{\vec{r}} = \vec{a} + 2\vec{\omega} \times \vec{v} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}}$$

Now

$$m\ddot{\vec{\rho}} = m(\ddot{\vec{R}} + \ddot{\vec{r}}) = \vec{F}$$

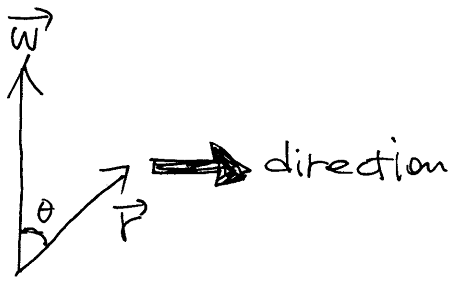
$$\Rightarrow m\ddot{\vec{r}} = \vec{F} - m\ddot{\vec{R}}$$

$$\Rightarrow \boxed{m\vec{a} = \vec{F} \underbrace{- m\ddot{\vec{R}} - 2m\vec{\omega}\times\vec{v} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - \dot{\vec{\omega}}\times\vec{r}}_{\text{Fictitious forces}}}$$

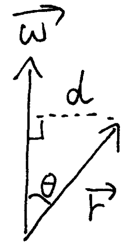
- $\vec{F}_{\text{Coriolis}} = -2m\vec{\omega}\times\vec{v}$: Coriolis force
- $\vec{F}_{\text{centrifugal}} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r})$: Centrifugal force

  $\perp \vec{\omega}$

  $\parallel$ (plane formed by $\vec{\omega}, \vec{r}$)



Norm: $|\vec{\omega}\times(\vec{\omega}\times\vec{r})| = \omega^2 \underbrace{r\sin\theta}_{= d}$

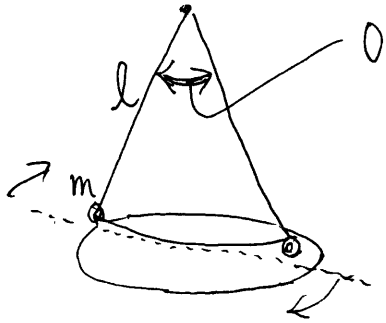


$$\boxed{|\vec{F}_{\text{centrifugal}}| = m\omega^2 d}$$

- $-m\ddot{\vec{R}}$: force of acceleration felt in a lift
- $-\dot{\vec{\omega}}\times\vec{r}$: present only if rotation is not constant.

Foucault's pendulum

- Proof that it is the earth that rotates, while the heavens are still.

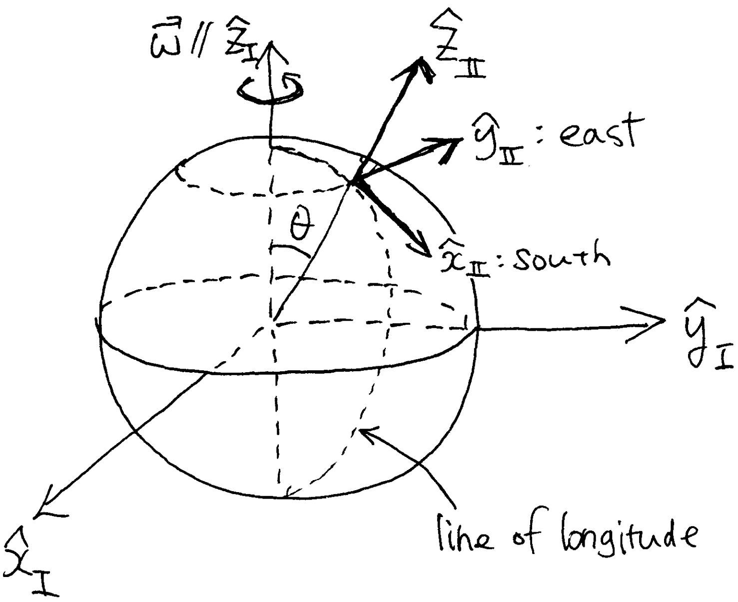


Oscillates at angular velocity $\boxed{\Omega = \sqrt{\frac{g}{l}}}$

Plane of orbit rotates clockwise (counter-clockwise) in northern (southern) hemisphere with angular velocity

$$\omega_{earth} \underbrace{\cos\theta}_{\text{colatitude}}$$

Take moving frame S_{II} aligned with local lines of longitude & latitude:

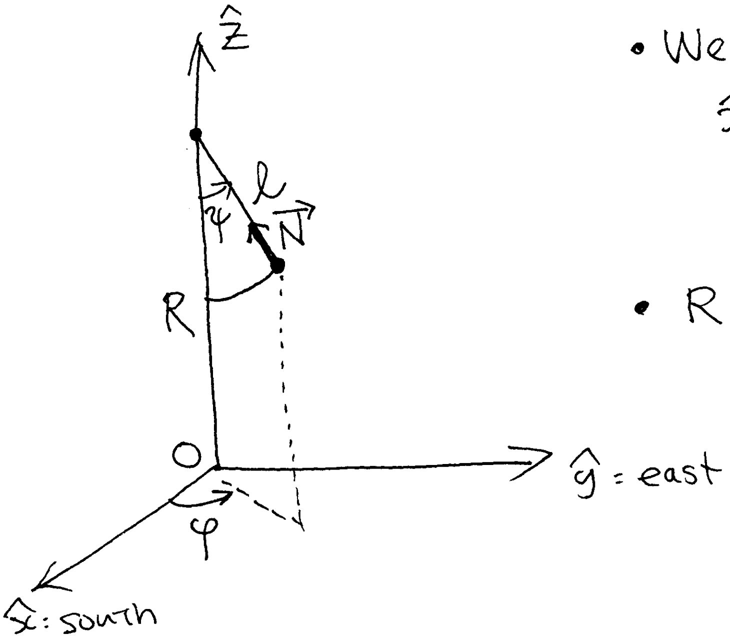


$\vec{\omega}$: angular velocity of earth

$$\omega = |\vec{\omega}| = \frac{2\pi}{24 \cdot 60 \cdot 60} = 7 \times 10^{-5}\, s^{-1} \sim 10^{-4}\, s^{-1}$$

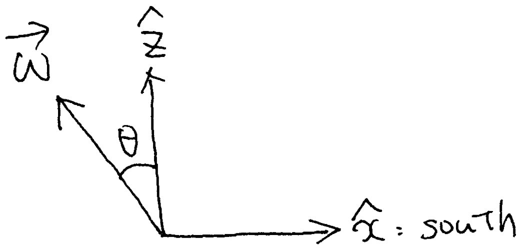
- Actually we take the origin of S_I & S_{II} to be the same. The above figure is only for explaining the orientation of S_{II} relative to S_I. So, $\vec{R} = 0$.

Here is our pendulum:



- We suppress "II"
  $\hat{x}_{II} \to \hat{x}$, $\hat{y}_{II} \to \hat{y}$,
  $\hat{z}_{II} \to \hat{z}$

- R = radius of earth
  $\sim 6000\,\mathrm{Km} = 6 \times 10^6\,\mathrm{m}$
  $\sim 10^7\,\mathrm{m}$



$$\vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} -\omega \sin\theta \\ 0 \\ \omega\cos\theta \end{pmatrix}$$

The EOM in the moving frame is, because $\ddot{\vec{R}} = \dot{\vec{\omega}} = 0$,

$$m\vec{a} = \vec{F} - 2m(\vec{\omega} \times \vec{v}) - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\vec{F} = \vec{N} + m\vec{g}$$

$\vec{a}, \vec{v}$: in the moving frame.

$$\vec{a} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

Writing out components,

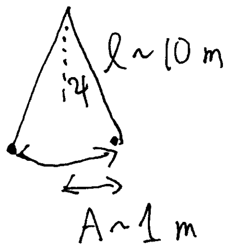
$$m\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = N\begin{pmatrix}-\sin\psi\cos\varphi\\ -\sin\psi\sin\varphi\\ \cos\psi\end{pmatrix} + \begin{pmatrix}0\\ 0\\ -mg\end{pmatrix} + \overbrace{2m\omega\begin{pmatrix}\cos\theta\,\dot{y}\\ -\cos\theta\,\dot{x}+\sin\theta\,\dot{z}\\ \sin\theta\,\dot{y}\end{pmatrix}}^{\text{Coriolis}}$$

$$+\underbrace{m\omega^2\begin{pmatrix}\cos\theta(\cos\theta x+\sin\theta z)\\ y\\ \sin\theta(\cos\theta x+\sin\theta z)\end{pmatrix}}_{\text{Centrifugal}}$$

z-motion

$$m\ddot{z} = N\cos\psi - mg + 2m\omega\sin\theta\,\dot{y} + m\omega^2\sin\theta(\cos\theta\,x-\sin\theta\,z)$$

Estimate each term when oscillation is small.



$\Rightarrow$ $v_{x\text{-}y} \sim 1\ \mathrm{m/s}$, $a_{x\text{-}y} \sim 1\ \mathrm{m/s^2}$

$v_z \sim 10^{-1}\ \mathrm{m/s}$, $a_z \sim 10^{-1}\ \mathrm{m/s^2}$

(see 6.10)

$\Rightarrow$ $|\ddot{z}| \sim a_z \sim 10^{-1}\ \mathrm{m/s^2}$

$|\omega\dot{y}| \sim \omega v_{x\text{-}y} \sim 10^{-4}\,\mathrm{s^{-1}}\cdot 1\ \mathrm{m/s} \sim 10^{-4}\ \mathrm{m/s^2}$

$|\omega^2 x| \ll |\omega^2 z| \sim \omega^2 R \sim (10^{-4})^2\, 10^7 \sim 10^{-1}\ \mathrm{m/s^2}$

On the other hand, $g \sim 10\ \mathrm{m/s^2}$.

So, these terms can be ignored. Furthermore, for $|\psi| \ll 1$,

$$N\cos\psi \approx N\left(1-\frac{\psi^2}{2}\right) \sim N$$

$$\Rightarrow\ 0 = N - mg \quad \Rightarrow \quad \boxed{N \approx mg}$$

x, y motion

$$
\begin{cases}
m\ddot{x} = -N\sin\psi\cos\varphi + \underbrace{2m\omega\cos\theta\,\dot{y}}_{①} + m\omega^2\cos\theta(\underbrace{\cancel{\cos\theta\cdot x}}_{②} + \underbrace{\sin\theta\cdot z}_{③}) \\
m\ddot{y} = -N\sin\psi\sin\varphi + 2m\omega(\underbrace{-\cos\theta\,\dot{x}}_{④} + \underbrace{\cancel{\sin\theta\,\dot{z}}}_{⑤}) + \underbrace{\cancel{m\omega^2 y}}_{⑥}
\end{cases}
$$

$① \sim ④ \sim \omega v_{x\text{-}y} \sim 10^{-4}\cdot 1 \sim 10^{-4}\ \mathrm{m/s^2}$ ✓ keep

$② \sim ⑥ \sim \omega^2 A \sim (10^{-4})^2\cdot 1 \sim 10^{-8}\ \mathrm{m/s^2}$ drop

$③ \sim \omega^2 R \sim (10^{-4})^2\, 10^{7} \sim 10^{-1}\ \mathrm{m/s^2}$ ✓ keep

$⑤ \sim \omega v_z \sim 10^{-4}\cdot 10^{-1} \sim 10^{-5}\ \mathrm{m/s^2}$ drop

Also,

$$N\sin\psi\cos\varphi \approx mg\sin\psi\cos\varphi = m\cdot\frac{g}{l}\cdot l\sin\psi\cos\varphi \sim m\Omega^2 x$$

$$N\sin\psi\sin\varphi \approx mg\sin\psi\sin\varphi = m\cdot\frac{g}{l}\cdot l\sin\psi\sin\varphi \sim m\Omega^2 y$$

$$\Omega^2 = g/l$$

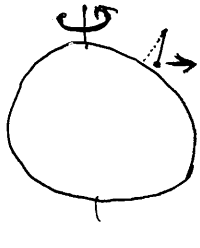
So,

$$
\begin{cases}
\ddot{x} = -\Omega^2 x + 2\omega_z\dot{y} + \underbrace{\Omega x_0}, \\
\ddot{y} = -\Omega^2 y - 2\omega_z\dot{x}
\end{cases}
$$

Where

$$\omega_z = \omega\cos\theta, \qquad x_0 = \frac{\omega^2 R}{\Omega^2}\cos\theta\sin\theta$$

- The Ωx_0 term can be absorbed by $x \to x - x_0$
  (This corresponds to shift in tilting due to centrifugal force)
  For $l = 10$ m, $x_0 \sim 3$ cm



Understanding this shift,

$$\boxed{\begin{aligned} \ddot{x} &= -\Omega^2 x + 2\omega_z \dot{y} \\ \ddot{y} &= -\Omega^2 y - 2\omega_z \dot{x} \end{aligned}}$$

If we define $\xi = x + iy$,

$$\boxed{\ddot{\xi} = -\Omega^2 \xi - 2i\omega_z \dot{\xi}}$$

Approximate sol'n

For $\omega_z \ll \Omega$, an approximate sol'n is

$$\boxed{\xi = e^{-i\omega_z t}\left(Ae^{i\Omega t} + Be^{-i\Omega t}\right)}$$

Proof:

If $\xi \propto e^{i\lambda t}$, then

$$-\lambda^2 = -\Omega^2 - 2i\omega_z i\lambda$$

$$\lambda^2 + 2\omega_z\lambda - \Omega^2 = 0$$

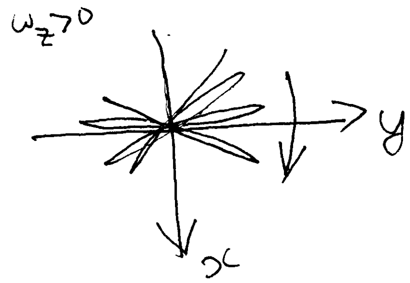
$$\lambda = -\omega_z \pm \sqrt{\omega_z^2 + \Omega^2} \approx -\omega_z \pm \Omega \qquad \Omega \gg \omega_z$$

So, the general solution is

$$\xi = Ae^{i(-\omega_z+\Omega)t} + Be^{i(-\omega_z-\Omega)t}$$

QED

The above sol'n tells us



$$\boxed{\xi = x + iy = e^{-i\omega_z t} \underbrace{\xi_0(t)}_{\text{Solution without influence of earth's rotation.}}}$$

So, $\xi = x + iy$ rotates about z (the local vertical) with a rate

$$\boxed{\omega_z = \omega_{Earth} \cos \underbrace{\theta}_{\text{colatitude}}}$$

- N. hemisphere $\Rightarrow$ $\theta \in (0, \frac{\pi}{2})$, $\cos\theta > 0$

  $\Rightarrow$ clockwise

- S. hemisphere $\Rightarrow$ $\theta \in (\frac{\pi}{2}, \pi)$, $\cos\theta < 0$

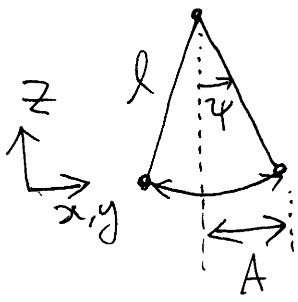
  $\Rightarrow$ anticlockwise

- Rotation is maximal at the north & south poles, not at the equator.

- One of the greatest proofs of the Earth's rotation (Foucault, 1851)

END OF LECTURES!

Estimate of various terms



$A \ll \ell$

In the first approximation, the pendulum is oscillating just as the standard pendulum. If oscillation is small, i.e. $|\varphi| \ll 1$, then

$$\varphi \approx a \cos(\Omega t + \alpha)$$

where a, α are constant, while

$$\Omega = \sqrt{\frac{g}{\ell}}.$$

The above is the oscillation in the angle φ.

The distance along x, y directions is

$$\ell\varphi \approx \ell a \cos(\Omega t + \alpha) \sim \ell a \equiv A : \text{amplitude along } x, y$$

The velocity along x, y directions is

$$v_{x\text{-}y} = \ell\dot{\varphi} \approx -\ell a \Omega \sin(\Omega t + \alpha) \sim \ell a \Omega = A\Omega$$

The acceleration is

$$a_{x\text{-}y} = \ell\ddot{\varphi} \approx -\ell a \Omega^2 \cos(\Omega t + \alpha) \sim \ell a \Omega^2 = A\Omega^2$$

$$\boxed{v_{x\text{-}y} \sim A\Omega, \quad a_{x\text{-}y} \sim A\Omega^2}$$

The motion along z is smaller as we can see from the figure. In equation,

$$z \approx \ell(1 - \cos\varphi) \approx \ell\varphi^2/2$$

$$\dot{z} \approx \ell\varphi\dot{\varphi} \sim \ell \cdot a \cdot a\Omega \sim \frac{A^2\Omega}{\ell}$$

$$\ddot{z} \approx \ell(\dot{\varphi}^2 + \varphi\ddot{\varphi}) \sim \ell a^2 \Omega^2 \sim \frac{A^2\Omega^2}{\ell}$$

So,

$$\boxed{v_z \sim \frac{A^2\Omega}{\ell}, \quad a_z \sim \frac{A^2\Omega^2}{\ell}}$$

these are down by $\frac{A}{\ell} \ll 1$ compared with $v_{x\text{-}y}$, $a_{x\text{-}y}$.

For example, if

$$\ell \sim 10\,m, \quad A \sim 1\,m$$

Then, using $g \sim 10\ m/s^2$,

$$\Omega = \sqrt{\frac{g}{\ell}} \sim \sqrt{\frac{10\ m/s^2}{10 m}} = 1\ s^{-1}$$

$$v_{x-y} \sim A\Omega \sim 1\ m/s$$

$$a_{x-y} \sim A\Omega^2 \sim 1\ m/s^2$$

$$v_z \sim \frac{A^2\Omega}{\ell} \sim \frac{1^2 \cdot 1}{10} \sim 10^{-1}\ m/s$$

$$a_z \sim \frac{A^2\Omega^2}{\ell} \sim \frac{1^2 \cdot 1^2}{10} \sim 10^{-1}\ m/s^2$$