USING THE LAGRANGE EQUATIONS

The Lagrange equations give us the simplest method of getting the correct equations of motion for systems where the natural coordinate system is not Cartesian and/or where there are constraints which reduce the number of degrees of freedom. Here are some simple examples of how we use the equations in practice. Often we do not need to solve the equations of motion explicitly in order to understand the nature of the motion. By clever use of the conserved quantities (symmetries) we can sidestep the tedious business of integrating differential equations numerically.

PLANETARY MOTION

Consider the simplest model of planetary motion in which we regard the Sun as fixed with mass M and the planet with mass m free to move under the Sun's gravitational attraction. Since the planet can move in any direction in three-dimensional space, it has three degrees of freedom. The next choice is the selection of a good set of generalised coordinates. Since the force on the planet depends only on its distance r from the Sun, it is pretty clear that spherical polar coordinates r, θ, ϕ provide a much better set of coordinates than do the Cartesian coordinates x, y, z. From the reference sheet on coordinate systems we have that the kinetic energy takes the form

$$T = \frac{1}{2}m\dot{\boldsymbol{r}}^2 = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) \quad .$$

The potential energy is

$$V(r) = -\frac{GMm}{r} \quad .$$

Thus the Lagrangian is

$$L = T - V = \frac{1}{2}m\dot{\boldsymbol{r}}^2 = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) + \frac{GMm}{r} \quad .$$

There will be three Lagrange equations, one corresponding to each of the three degrees of freedom.

r eqn.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 - mr\sin^2\theta\dot{\phi}^2 + \frac{GMm}{r^2} = 0 \quad . \tag{1}$$

θ eqn.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}(mr^2\dot{\theta}) - mr^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0 \quad . \tag{2}$$

ϕ eqn.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = \frac{d}{dt}(mr^2\sin^2\theta\dot{\phi}) = \frac{d}{dt}p_\phi = 0 \quad , \tag{3}$$

where the last equation tells us that the generalised momentum p_ϕ is conserved, reflecting the fact that ϕ is a cyclic or ignorable coordinate (rotational symmetry).

You should become accustomed to carrying out the steps above for any problem. At this point however, we must start to think instead of simply turning the handle of deriving Lagrange equations. If necessary, we can now solve the Lagrange equations numerically by resorting to computers. However, rarely do we need to know such detailed information. More often it is sufficient that we can determine qualitative features of the motion such as whether or not it is bounded. For qualitative purposes we should instead make full use of the conserved quantities in the problem. In particular, if there are conserved momenta, we can use them to eliminate variables from the energy equation thus reducing the complexity of the problem. In the present case we have two conserved quantities, namely the momentum $p_\phi = mr^2 \sin^2\theta\,\dot{\phi}$, and the energy $E = p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - L = T + V$. From p_ϕ we solve for $\dot{\phi}$,

$$\dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta} \quad , \tag{4}$$

where p_ϕ is constant, and substitute in the energy equation

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - \frac{GMm}{r} \quad ,$$

to get

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{p_\phi^2}{2mr^2\sin^2\theta} - \frac{GMm}{r} = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V_{eff}(r,\theta) \quad .$$

Here I have observed that the last piece of the kinetic energy now looks exactly like a potential energy since it is a function only of r and θ and does not depend on any velocities. Therefore I have included it with the true potential energy to define an *effective* potential

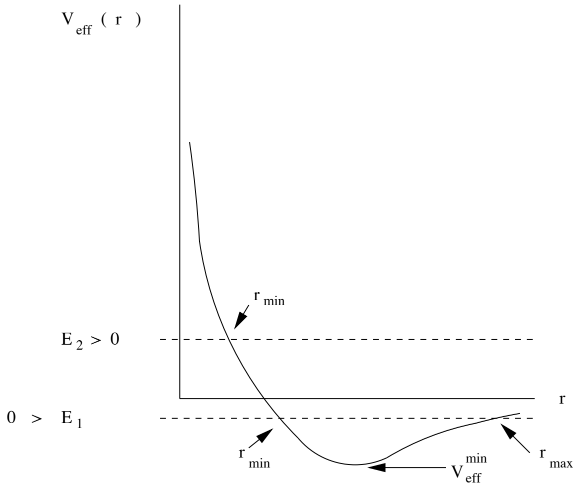
$$V_{eff}(r,\theta) = \frac{p_\phi^2}{2mr^2\sin^2\theta} - \frac{GMm}{r} \quad .$$

Already we can learn something. So long as there is a non-zero angular momentum value p_ϕ, the first term in the effective potential blows up to positive infinity as $r \to 0$. However, since the energy E is constant, at some point we would violate the energy equation. Thus the motion of the particle is such that there is a least distance r_{min} it can reach from the Sun, provided only that it has some non-zero angular momentum.

We can say even more if we look at the Lagrange equation (2) for θ. There are obvious special solutions for $\theta = constant$ where the constant values are $\theta = 0, \pi$ or $\theta = \pi/2$. The first two of these correspond to zero angular momentum p_ϕ and to the planet falling straight into the sun along the polar axis. The third value, $\theta = \pi/2$ corresponds to motion in a plane at right angles to the polar direction. For this constant value of $\theta = \pi/2$, $\dot{\theta} = 0$ the equations above simplify even more to become

$$E = \frac{m}{2}\dot{r}^2 + \frac{p_\phi^2}{2mr^2} - \frac{GMm}{r} = \frac{m}{2}\dot{r}^2 + V_{eff}(r) \quad .$$

Now the problem is reduced to one concerning a single variable r. If we plot the effective potential as a function of r, we see that it starts from $+\infty$ at $r = 0$, falls rapidly and becomes negative and then slowly vanishes as $r \to \infty$. To satisfy the energy conservation equation the value of $V_{eff}(r)$ at point r must always be less than or equal to E. Thus for a negative energy E_1 such that $0 > E_1 > V_{eff}^{min}$, the physically allowed region is finite, bounded by a least r_{min} and a greatest r_{max}. While for a positive energy $E_2 \geq 0$, the motion has a least distance r_{min} but it is unbounded at infinite distance allowing the planet to escape to infinite distance. The precise values of r_{min} and r_{max} depend on the values of the conserved quantities E and p_ϕ which in turn are determined by the initial values of position and velocity for the planet.



THE SPHERICAL PENDULUM

As a second example, we consider the spherical pendulum which consists of a point mass m connected to a pivot point by a light massless rigid rod of length a. We then set it in motion by striking it at random so that in the subsequent motion it moves arbitrarily provided that its distance from the pivot point remains constant. In effect the mass moves on the surface of a sphere which is two-dimensional so there are two degrees of freedom. The best coordinate system to describe this moving mass is a spherical polar coordinate system r, θ, ϕ with an origin at the pivot point so that the constraint of constant distance becomes simply $r = a = constant$ and θ, ϕ are the two generalised coordinates describing the two independent degrees of freedom. Since r is constrained to be constant, $\dot{r} = 0$, it drops out of the problem completely and the Lagrangian becomes

$$L = T - V = \frac{1}{2} m \dot{\boldsymbol{r}}^2 = \frac{m}{2} \left(a^2 \dot{\theta}^2 + a^2 \sin^2 \theta \dot{\phi}^2 \right) - mga \cos\theta \quad .$$

Now there are only two Lagrange equations, one for θ and one for ϕ.

θ eqn.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}(mr^2\dot{\theta}) - mr^2 \sin\theta \cos\theta \, \dot{\phi}^2 - mga \sin\theta = 0 \quad .$$

ϕ eqn.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = \frac{d}{dt}(mr^2 \sin^2\theta \dot{\phi}) = \frac{d}{dt} p_\phi = 0 \quad .$$

Thus there is again a conserved generalised momentum p_ϕ and again there is a conserved energy $E = p_\theta \dot{\theta} + p_\phi \dot{\phi} - L = T + V$. Just as in the earlier example we can simplify by eliminating $\dot{\phi}$ in favour of p_ϕ as in equation (4) . If we substitute in the energy equation

$$E = \frac{m}{2} \left(a^2 \dot{\theta}^2 + a^2 \sin^2 \theta \dot{\phi}^2 \right) + mga \cos\theta \quad ,$$

we get

$$E = \frac{m}{2}a^2\dot{\theta}^2 + \frac{p_\phi^2}{2ma^2\sin^2\theta} + mga\cos\theta = \frac{m}{2}a^2\dot{\theta}^2 + V_{eff}(\theta) \quad .$$

In this case, the effective potential is a function of θ only so it looks like a one dimensional problem again with a single variable to think about. If we now plot

$$V_{eff}(\theta) = \frac{p_\phi^2}{2ma^2\sin^2\theta} + mga\cos\theta \quad ,$$

as a function of θ between $\theta = 0$ and $\theta = \pi$ we see that the first term blows up to $+\infty$ near both endpoints $\theta = 0, \pi$ while the second term is a simple finite cosine curve. At fixed energy E we again find the physically allowed region as those values of θ such that $E \geq V_{eff}(\theta)$. Thus we can prove the motion is bounded without solving the Lagrange equations explicitly. The magnitude of the bounds θ_{max} and θ_{min} depends on the initial conditions at time $t = 0$ which will fix the values of the conserved quantities E and p_ϕ.